Stochastic Signals and Systems

Spectral theory, I.

Lecture 3.

Completely regular process. Review.

Let (y_n) be a completely regular process, i.e.

$$H^y_{-\infty} = \bigcap_{m \geq 0} H^y_{-m} = \{0\}.$$

Then for (y_n) we have:

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k}$$

with $h_0 = 1$, and $\sum_{k=0}^{\infty} h_k^2 < \infty$. Moreover $H_n^e = H_n^y$ for all n.

The problem of prediction revisited

Let (y_n) be a completely regular process

$$y_n = \sum_{k=0}^{\infty} h_k e_{n-k} = e_n + \sum_{k=1}^{\infty} h_k e_{n-k},$$

where $h_0 = 1$, and (e_n) is the innovation process of y.

Then the LSQ predictor of y_n is given by

$$\widehat{y}_n = \sum_{k=1}^{\infty} h_k e_{n-k}.$$

The problem: express $\widehat{y}$ in terms of y - instead of e!

Simplify the problem: assume that (y_n) is an MA(r) process:

$$y_n = \sum_{k=0}^{r} c_k e_{n-k}, \quad c_0 = 1. \tag{1}$$

A useful tool: the *backward shift operator*:

$$(y_n) \quad \mapsto \quad \big((q^{-1}y)_n\big), \qquad (q^{-1}y)_n = y_{n-1}.$$

Introduce a **polynomial** of q^{-1} as

$$H(q^{-1}) = 1 + c_1 q^{-1} + \ldots c_r q^{-r} = \sum_{k=0}^{r} c_k q^{-k}.$$

Then the MA process (1) can be rewritten as

$$\boxed{y = H(q^{-1})e}$$

$$y = H(q^{-1})e.$$

The LSQ predictor is

$$\widehat{y}_n = y_n - e_n.$$

Thus, the process $(\widehat{y}_n)$ can be defined this way:

$$\widehat{y} = \left(H(q^{-1}) - 1\right) e.$$

To express e via y, a formal procedure is to invert $y = H(q^{-1})e$ as

$$e = H^{-1}(q^{-1})y.$$

What is the interpretation of the operator $H^{-1}(q^{-1})$?

An example: (y_n) is an MA(1) process

$$y_n = e_n + c e_{n-1}. \qquad (2)$$

Then

$$e_n = -c e_{n-1} + y_n.$$

Iterating this equation we get

$$e_n = \sum_{k=0}^{\infty} (-c)^k y_{n-k}. \qquad (3)$$

It is well defined, if $|c| < 1$. (3) is the inverse of (2). Thus

$$H(q^{-1}) = 1 + c q^{-1}, \qquad H^{-1}(q^{-1}) = \sum_{k=0}^{\infty} (-c)^k q^{-k}$$

The situation becomes much more complicated for higher order MA models. The interpretation of $H^{-1}(q^{-1})$ needs extra care.

A detour: the z transform, I.

An excursion to the theory of linear time invariant (LTI) systems:

$$y_n = \sum_{k=0}^{n} h_k u_{n-k}, \qquad n \geq 0.$$

Here $u = (u_n)$ is the *input process*, $y = (y_n)$ is the *output process*, $n \geq 0$.

The coefficients h_k are the *impulse responses*.

Assume that

$$\sum_{k=0}^{\infty} |h_k| < \infty.$$

Let $u = (u_n)$ be bounded. Then $y = (y_n)$ will also be bounded.

Define for $|z| > 1$:

$$U(z^{-1}) = \sum_{k=0}^{\infty} u_k z^{-k}, \quad Y(z^{-1}) = \sum_{k=0}^{\infty} y_k z^{-k}, \quad H(z^{-1}) = \sum_{k=0}^{\infty} h_k z^{-k}.$$

The power series are absolute convergent for $|z| > 1$.

Then we have a simple multiplicative description of our LTI:

$$Y(z^{-1}) = H(z^{-1})\ U(z^{-1}).$$

Extend it to **two sided** processes: neither $|z| > 1$, nor $|z| < 1$ would do.

The only option is to try $|z| = 1$, i.e. $z = e^{i\omega}$.

Fourier methods for w.s.st. processes, I.

Thus we are led to the formal objects: $\sum_{n=-\infty}^{\infty} y_n e^{-in\omega}, \qquad \omega \in [0, 2\pi).$

The aim of spectral theory is to give a meaning to these formal objects.

Let (y_n) be a w.s.st. process. We can ask if the finite Fourier series

$$\xi_N := \sum_{n=-N}^{N} y_n e^{-in\omega}$$

has a limit in any sense? ($\mathrm{E}\xi_N = 0$.)

For a start ask if the following sequence has a limit?

$$\lim_{N\to\infty} \mathrm{E} \left| \sum_{n=-N}^{N} y_n e^{-in\omega} \right|^2 = ?$$

Consider a fix N, and compute:

$$\mathrm{E}\left|\sum_{n=-N}^{N} y_n e^{-in\omega}\right|^2 = \mathrm{E}\left(\sum_{n=-N}^{N} y_n e^{-in\omega} \sum_{m=-N}^{N} y_m e^{+im\omega}\right) =$$

$$= \sum_{n=-N}^{N} \sum_{m=-N}^{N} \mathrm{E}\left(y_n y_m\right) e^{-i(n-m)\omega}$$

As $\mathrm{E}\left(y_n y_m\right) = r(n-m)$, introduce a new variable $\tau = n - m$. Thus

$$\mathrm{E}\left|\sum_{n=-N}^{N} y_n e^{-in\omega}\right|^2 = \sum_{\tau=-2N}^{2N} r(\tau)\ e^{-i\omega\tau}\left(2N + 1 - |\tau|\right). \tag{4}$$

Fourier methods for w.s.st. processes.

Defining

$$s_n(\omega) = \sum_{\tau=-n}^{+n} r(\tau) e^{-i\omega\tau},$$

we can write

$$\sum_{\tau=-2N}^{2N} r(\tau)\ e^{-i\omega\tau}\,(2N+1-|\tau|) = \sum_{n=0}^{2N} s_n(\omega).$$

This is the *Cesaro-sum* of a Fourier series: *the sum of partial sums*.

$$\Longrightarrow \qquad \mathrm{E}\left|\sum_{n=-N}^{N} y_n e^{-in\omega}\right|^2 = \sum_{n=0}^{2N} s_n(\omega)$$

At this point let us make the *assumption* that

$$\sum_{\tau=-\infty}^{+\infty} r^2(\tau) < +\infty. \tag{5}$$

Assumption (5) implies that

$$\lim_{n\to\infty} \sum_{\tau=-n}^{+n} r(\tau)e^{-i\omega\tau} = \sum_{\tau=-\infty}^{+\infty} r(\tau)e^{-i\omega\tau} =: f(\omega)$$

is well-defined, $f(\omega) = \lim_{n\to\infty} s_n(\omega)$ in $L_2([0, 2\pi], d\omega)$.

Fejér's theorem: The sum also converges to $f(\omega)$ in the Cesaro sense:

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} s_n(\omega) = f(\omega) \qquad \text{a.s..}$$

Fourier methods for w.s.st. processes, III.

Proposition

Under condition $\sum r^2(\tau) < +\infty$

$$\lim_{N\to\infty} \frac{1}{2N+1} \mathrm{E} \left| \sum_{n=-N}^{N} e^{-in\omega} y_n \right|^2 = f(\omega) \geq 0$$

exists a.s. on $[0, 2\pi)$ *w.r.t. the Lebesgue-measure, where*

$$f(\omega) = \sum_{\tau=-\infty}^{+\infty} r(\tau) e^{-i\omega\tau}.$$

Herglotz's theorem, special form.

We immediately get a special form of the celebrated Herglotz's theorem:

Proposition

Under condition $\sum r^2(\tau) < +\infty$ *we have*

$$r(\tau) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega\tau} f(\omega) d\omega$$

with some $f(\omega) \geq 0$, $f(\omega) \in L_2[0, 2\pi)$. *In particular*

$$r(0) = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) d\omega.$$

This is the ***spectral density function*** *of the w.s.st process* y.

Herglotz's theorem, example.

Let $(y_n) = (e_n)$ be an *orthogonal* process.

Then the autocovariances are:

$$r^e(0) = \sigma^2, \qquad \text{and} \quad r^e(\tau) = 0 \qquad \text{for} \quad \tau \neq 0.$$

In this case

$$f(\omega) = \sigma^2, \qquad \forall \omega \in [0, 2\pi).$$

The spectral density function is constant.

Exercise. (HW) Let y be an MA(1) process: $y_n = e_n + c e_{n=1}$. Compute the spectral density function of y.