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# Combinatorial Methods

$H$ a set system, $H = \{H_1, H_2, H_3 \dots H_n\}$

**Def**: Line graph of $H$, $L(H)$ has $n$ vertices $(v_1, v_2, v_3, \dots v_n)$ and $v_i v_j$ edge $\Leftrightarrow H_i \cap H_j = \phi$, $i \neq j$

Let $G = (V, E)$ be a graph ($V$: vertex set, $E$: edge set)

**Def**: Independent set: $S \subseteq V$, no two $v, w \in S$ are adjacent
e.g.: $\phi$, single vertex, non-adjacent pair etc.

**Def**: Independence number $(\alpha(G))$: max size of an independent set in $G$

**Def**: Complete subgraph: any 2 of its vertices are adjacent
e.g.: $\phi$, single vertex, edge, ...

**Def**: Clique number $(\omega(G))$: max # of vertices in a complete subgraph of $G$

**Def**: Proper vertex coloring: $\varphi: V \to C$ (set of colors, say $C = \{1, 2, 3, \dots k\}$) such that $\varphi(v) \neq \varphi(w)$ whenever $vw \in E$

Chromatic number $(\chi(G))$: smallest $k$ such that $G$ has a proper coloring with $k$ colors

**Prop**: $\forall G$: $\chi(G) \geq \omega(G)$

**Prop**: $\chi(G)$ = min # of independent sets whose union is $V(G)$
- Each color class in a proper coloring is independent
- The vertices in an independent sets can be monochromatic

**Def**: Clique cover: collection of complete subgraphs whose union is $V(G)$

**Def**: Clique covering number $(\Theta(G))$: min # of complete subgraphs in a clique cover

**Prop**: $\forall G$, $\Theta(G) \geq \alpha(G)$

**Def**: Complementary graph or complement of $G$ $(\overline{G})$: $vw \in E(G) \Leftrightarrow vw \notin E(G)$ and $V(G) = V(\overline{G})$

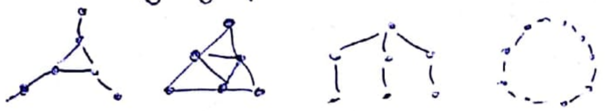
**Prop**: $\overline{(\overline{G})} = G$, $\alpha(G) = \omega(\overline{G})$, $\alpha(\overline{G}) = \omega(G)$
$\chi(G) = \Theta(\overline{G})$, $\chi(\overline{G}) = \Theta(G)$

~~Unsupported Personality:~~ UNKNOWN

| $\mathcal{H}$ | $\mathcal{L}(\mathcal{H})$ |
|---|---|
| set | vertex |
| intersect | adjacent |
| disjoin | non-adjacent |
| intersecting subsystem | complete subgraphs |
| decomp. into —"— | clique cover |
| $k(\mathcal{H})$ | ~~$\Theta(L(\mathcal{H}))$~~ $\Theta(L(\mathcal{H}))$ |
| packing / matching | independent set |
| $\nu(\mathcal{H})$ | $\alpha(L(\mathcal{H}))$ |
| decomposition into packings | proper coloring |
| $q(\mathcal{H})$ | $\chi(L(\mathcal{H}))$ |

Def: Interval graph: intersection graph of some interval system

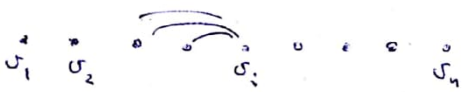
Remark: Not every graph is an interval graph



these are not line graphs of any interval system

Theorem: $\alpha, \omega, \chi, \Theta$ can be computed in polinomial time on interval graphs

$\Rightarrow$ Efficiently computable general upper bound on $\chi(G)$



given fixed order, denote

$\bar{d}(v_i)$: # of neighbours $\to v_j$ of $v_i$ : $j<i$

Obs: If $k > \max_{1\le i\le n} \bar{d}(v_i)$ then $G$ has a proper coloring with $k$ colors

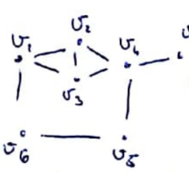
First-Fit (greedy) coloring:

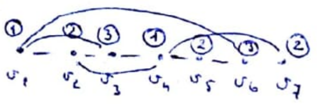
~~for~~ for $i = 1 \dots n$ assign the smallest possible color from $1, 2, 3, \dots$ to $v_i$

If $\bar{d}(v_i) < k$ then at most $k-1$ colors are excluded from $v_i$ in the $i^{th}$ step $\Rightarrow \exists$ free color for $v_i$

Def: Coloring number ($col(G)$): $col(G) = \min_{\substack{v_1 \dots v_n \\ \text{vertex orders}}} \max_{1\le i\le n} (\bar{d}(v_i)+1)$

Theorem: $col(G)$ is computable in polynomial time





Proof: algorithm:

Let $v_n$: vertex of min degree in $G$

$v_{n-1}$: ——"—— $G - v_n$

$\vdots$

$v_i$: ——"—— $G - \{v_{i+1}, v_{i+2} \dots v_n\}$

just universal upper bound for $\chi$, because e.g. $\square$ performs better on First-Fit

2018.03.04

# Combinatorial Methods #4

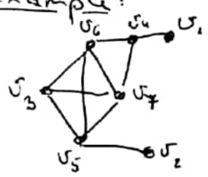
<u>Theorem</u>: If $G$ is chordial, then $\alpha(G) = \Theta(G)$ and $\alpha, \Theta$ can be determined in polynomial time

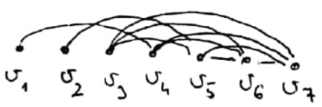
<u>Proof</u>: Find simplicial order $v_1, \dots v_n$ .

- put first vertex into set $S$
- put the vertex and its neighbors as a complete subgraph into $K$, which will be a clique cover
- remove all these vertices

repeat

<u>Example</u>:





| step | $S$ | $K$ | del |
|---|---|---|---|
| 1. | $v_1$ | $\{v_1, v_4\}$ | $\{v_1, v_4\}$ |
| 2. | $v_2$ | $\{v_2, v_5\}$ | $v_2, v_5$ |
| 3. | $v_3$ | $\{v_3, v_6, v_7\}$ | $v_3, v_6, v_7$ |

$\alpha = |S| = 3 = \Theta = |K|$

→ $S$ is independent (all neighbors of selected vertex are deleted)

→ $K$ is clique cover (vertices are deleted only when they are already covered)

→ Members of $K$ are complete because we have simplicial order and choose first vertex

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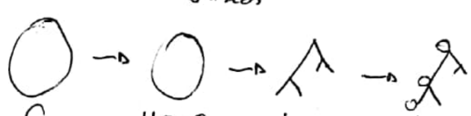
## ~~Thm~~ Treewidth

cordial ↔ subtrees




<u>Definition</u>: Tree decomposition of $G=(V,E)$: $(T, \xi)$, where $T$ is a rooted tree with nodes $x_1, x_2, \dots x_k$, and $\xi = (S_1, \dots S_k)$ is set of subsets, where $\forall S_i \subseteq V(G)$: $\forall i = 1 \dots k$ and the following properties are true:

(i) $\forall v \in V(G)\ \exists i: v \in S_i$

(ii) $\forall v, w \in E(G)\ \exists i: v, w \in S_i$

(iii) If $v \in S_{i_1}$ and $v \in S_{i_2}$ then $v \in S_i$ for all $i$ such that $x_i$ is on the $x_{i_1} - x_{i_2}$ path of $T$

<u>Definition</u>: Width of $(T, \xi)$ is $\max |S_i| - 1$

<u>Definition</u>: Treewidth of $G$ ($tw(G)$) is the minimal width among all the representations $(T, \xi)$

<u>Fact</u>: $tw(G) = \min\limits_{\substack{H \\ \text{chordial} \\ G \subseteq H}} (\omega(G) - 1)$

<u>Definition</u>: A tree decomposition $(T, \xi)$ is a <u>nice</u> tree decomposition, if $T$ is a rooted binary tree (each node has at most 2 children), and only the following four types of nodes occure:

- Leaf: has no children
- Introduce: $x_i$ — $x_j$: $S_i = S_j \cup \{v\}$
- Forget: $x_i$ — $x_j$: $S_i = S_j \setminus \{v\}$
- Join: $x_i$ — $x_{j_1}, x_{j_2}$: $S_i = S_{j_1} = S_{j_2}$

# Combi #6

page 1.

2018.03.20.

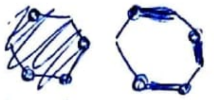
**Matching** $G=(V,E)$

~~$G=(V,E)$~~, matching number: $\nu(G)$ : max # of disjoint edges

transversal number, $\tau(G)$: min # of vertices meeting all edges

$\forall G: \tau(G) \geq \nu(G)$

Odd cycles: $C_{2t+1}$



$\nu: t$

$\tau: t+1$

Determine $\tau$ and $\nu$ on graphs:

$\tau$: NP-complete

$\nu$: polynomial time

**Def**: Biparted graph: $\chi \leq 2$

V can be partitioned into 2 independent sets

Biparted graphs can be recognized in linear time (e.g. by breadth-first search)

G is bipartite $\Leftrightarrow$ each $L_i$ is independent


→ If some $L_i$ is not independent, then G contains an odd cycle

Otherwise $\left.\begin{array}{l} L_0 \cup L_2 \cup L_4 \ldots \\ L_1 \cup L_3 \cup L_5 \ldots \end{array}\right\}$ independent sets

Fact: A graph is bipartite $\Leftrightarrow$ contains no odd cycles

**König's theorem**: If G is bipartite, then $\tau(G)=\nu(G)$, and a min. transversal and max matching can be found in polynomial time

**Proof**: Algorithm:

- find maximal matching (under inclusion)

**Optimizing matchings:**

$G=(V,E)$, $V=A\cup B$, M is non-extendable matches


$X\subset A$, $Y\subset B$ → vertex subsets not covered by M


**Def**: Alternating path: every second edge is in M

**Def**: Augmenting path: alternating path from X to Y

Augmenting → odd length

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More edges outside M, than in M

→ Switch: delete its M-edges from M, and insert its non-M edges into M

→ Remains matching

→ Matching size increased

Do as long as possible

# Combi # 6



For the case $|A|=|B|$

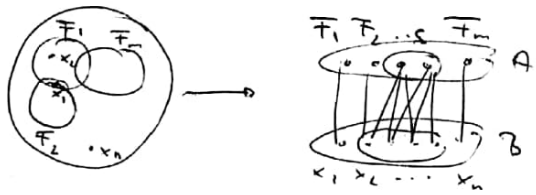
Def: Perfect matching (also called 1-factor): matching that covers the entire vertex set

Hall: A bipartite graph with $|A|=|B|$ has a perfect matching $\Leftrightarrow$ Hall's Condition is satisfied

Alternative formulation: $\mathcal{F}$ set system over $X$ ($\mathcal{F}=\{F_1, F_2 \dots F_m\}$, $X=\{x_1, x_2, \dots x_n\}$)

Def: System of Distinct Representatives: Subset $(x_{i_1}, x_{i_2}, \dots x_{i_m})$ with $m$ elements such that $x_{i_j} \in F_j$ : $\forall j = 1 \dots m$ $\longrightarrow$ SDR

Hall's Theorem: A set system has an SDR $\Leftrightarrow$ for every $k$ the union of any $k$ set contains at least $k$ elements




$x_i F_j$ is an edge if $x_i \in F_j$

If $S$ is a subfamily of $\mathcal{F}$, then in the bipartite graph $N(S)$ is the union of the members of $S$

SDR: matching that covers $\{F_1 \dots F_m\}$

$\Downarrow$

$\exists$ SDR $\Leftrightarrow$ $A$ is matchable into $B$ $\Leftrightarrow$ Hall's Condition holds

Def: A graph is d-regular, if every vertex has degree $d$

Def: A graph is regular, if d-regular for some $d$)

Theorem: Every regular bipartite graph of degree $\geq 1$ has a perfect matching

Proof: Suppose d-regular

\# of edges $S - N(S)$:

$d \cdot |S| = \quad \leq d \cdot |N(S)|$ $\Rightarrow$ Hall's condition holds

# Combi # 7

page 2.

If no augmenting paths:

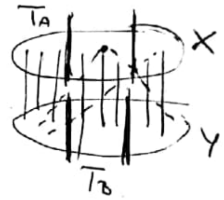
- → If $X = \emptyset$, then $|A| = |M|$ and $A$ meets all edges
- → If $Y = \emptyset$, then $|B| = |M|$ and $B$ meets all edges

  ⇒ $\tau \le |A| = |M| \le \nu \le \tau$ equality throughout
- → If $X, Y \ne \emptyset$, we shall find a traversal $T$ of size $|M|$

  $T_B$ := set of vertices in $B$ reachable from $X$ along alternating paths

  $T_A$ := the $A$-vertices of those $M$-edges which do not meet $T_B$

  $T := T_A \cup T_B$




Claim: $T$ is a ~~traversal~~ transversal

no edges between $X - Y$ (maximal from beginning)

$X - (B \setminus (T_B \cup Y))$ → def of $T_B$

$A \setminus (T_A \cup X) - Y$ → such edge would yield an augmenting path

$A \setminus (T_A \cup X) - B \setminus (T_B \cup Y)$ → this would contradict the def. of $T_B$

⇒ $\tau \le |T| = |M| \le \nu \le \tau$

$M$ is the ~~lg~~ largest matching, $T$ is the smallest transversal

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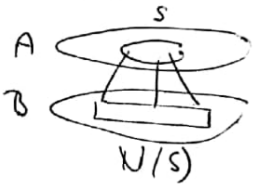



$S \subseteq A$

$N(S)$: set of vertices in $B$ which have at least one neighbor in $S$

Fact: if $\exists$ matching covering all vertices of $A$, ~~then~~ then $|N(S)| \ge |S|$

Hall's Condition:

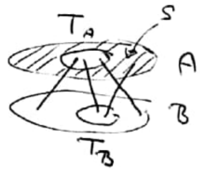
$\forall S \subseteq A$, $|N(S)| \ge |S|$

Hall's Marriage Theorem:

In bipartite graph, $G = (V, E)$ with vertex classes $A$ and $B$, there exists a matching covering $A$ if and only if Hall's condition is satisfied

(In case "$A$ is matchable into $B$")

Proof: from Kőnig's theorem:




Suppose for ~~and~~ contradiction: $\nu(G) < |A|$,

then $\tau(G) < |A|$

Let $T = T_B \cup T_A$ transversal, $|T| = \tau$

Consider $S = A \setminus T_A$ → $N(S) \subseteq T_B$

$|N(S)| \le |T_B| = |T| - |T_A| < |A| - |T_A| = |S|$

⇒ $S$ violates Hall's condition

∎

# Comb: #8



- <u>Chromatic index:</u> $G = (V, E)$

  <u>Proper edge coloring:</u> $\phi : E \to \mathbb{N}$ such that $\phi(e) \neq \phi(e')$ $\forall e, e' \in E$ such that $e \cap e' \neq \emptyset$

  <u>Def:</u> Chromatic index, $\chi'(G)$ smallest k such that G has a proper edge coloring with k colors

  <u>Remark:</u> $\chi'(G) \geq \Delta(G)$ ← maximum degree ⟶ good approximation

- <u>Vizing</u> <u>Theorem:</u>

  $\forall G$, $\chi'(G) \leq \Delta(G) + 1$

  <u>Remark:</u> $\chi'(G) = \chi(L(G))$

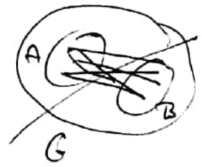
# Combi #9

## Max-cut:

Given $G = (V, E)$

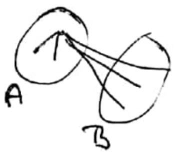
Unweighted version: determine the max # of edges in a bipartite subgraph.

denote max: $mc(G)$

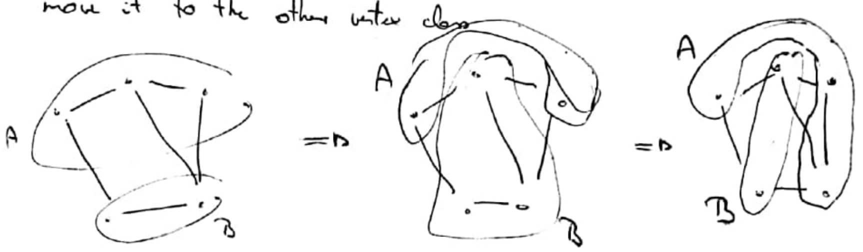



**Thm**: $\forall G: mc(G) \geq \frac{1}{2}|E|$

1st proof: local optimum




If $\exists$ vertex with more neighbors in its class than in the others, move it to the other vertex class




$mc(G) \geq 5$

In each step the size of cut increases $\Rightarrow$ terminates if $v$ has degree $d(v)$

$\Rightarrow$ in local optimum it contributes to the cut with $\frac{1}{2}d(v) \Rightarrow |cut| \geq \frac{1}{2}\sum_v\left(\frac{1}{2}d(v)\right) = \frac{|E|}{2}$

2nd proof ("online")

Choose vertex order $v_1, v_2, v_3 \ldots v_n$

$A = B = \emptyset$

for $i = 1 \ldots n$

put $v_i$ into B if it has more neighbors in A, and put into A otherwise

| A | $v_1$ | $v_4$ $v_5$ |
|---|---|---|
| B | $v_2$ $v_3$ | |

$d^-(v_i)$ degree to earlier neighbors

$|cut| \geq \frac{1}{2}\sum_{v_i} d^-(v_i) = \frac{1}{2}|E|$

3rd proof (probabilistic)

Let $P_r(v \in A) = \frac{1}{2}$

$e \in cut \Leftrightarrow v \in A, u \in B$ or $u \in A, v \in B$ $\quad (e = uv)$

$P_r(e \in cut) = \frac{1}{2}$

$E(|cut|) = \frac{1}{2}(E)$

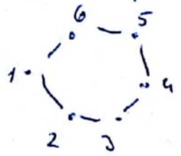
expected value

expectation is between min and max $\Rightarrow mc(G) \geq E(|cut|) \geq \frac{1}{2}|E|$

## Forwarding index

vertices: $v_1 \dots v_n$

define a path $P_{i,j}$ : $\forall i \neq j$ from $v_i$ to $v_j$




if multiple shortest path: may choose freely → e.g.: $P_{14} = 1234$, $P_{14} = 4561$

$load(v_k) = \#$ paths $P_{ij}$, where $v_k$ is an internal vertex

⟹ want to minimize longest $load(v_k)$

$P_{ij}$ has $\geq (d(v_i, v_j) - 1)$ internal vertices
distance between $v_i, v_j$

Total load: $\geq \sum_{i \neq j} (d(v_i, v_j) - 1) = n(n-1) - 2|E|$

Denote $\xi(G)$ minimum of longest ~~leg~~ load over all possible path systems (routings)

$\xi(G)$ is called forwarding index of $G$ ⟹ $\xi(G) \geq n - 1 - \frac{2|E|}{n}$

<u>Theorem</u>: There exist graphs with $n$ vertices and $\sim \frac{1}{2} n^{3/2}$ edges such that $\xi(G) \sim n$

<u>Proof</u>: suppose $n = q^2 + q + 1$, $q$ is a prime power

$GF(q)$: Galois field of order $q$

→ If $q$ is prime, then addition/multiplication can be computed modulo $q$

Represent the vertices with "homogeneous triples": $(a, b, c) \neq (0,0,0)$, $a, b, c \in GF(q)$

→ $(\lambda a, \lambda b, \lambda c)$ means the same vertex for all $\lambda \in GF(q) \setminus \{0\}$

e.g.: $q = 5$ → e.g. $(1,2,4)$, $(2,4,3)$, $(3,1,2)$, $(4,3,1)$

$\#$ of triples $\neq (0,0,0)$: $\frac{q^3 - 1}{q - 1} = q^2 + q + 1$

edges: $(a, b, c)$ and $(a', b', c')$ adjacent if $aa' + bb' + cc' = 0$ in $GF(q)$

→ e.g.: $(1,2,4)$, $(1,3,2)$ → $\underbrace{1 \cdot 1}_{1} + \underbrace{2 \cdot 3}_{1} + \underbrace{4 \cdot 2}_{3} = 0$

degree of vertex $(a,b,c)$: $\#$ of homogeneous solutions of $ax + by + cz = 0$

dimension of solution space $= 2$

nonzero ~~neighbors~~: $q^2 - 1$, neighbors: $\frac{q^2 - 1}{q - 1} = q + 1$

except when $a^2 + b^2 + c^2 = 0$ → all vertices are $q$ or $q + 1$

if 2 vertices are nonadjacent then they have a common neighbor $u(a,b,c)$, $v(a',b',c')$

neighbors $(x, y, z)$

$$\left.\begin{aligned} ax + bx + cz &= 0 \\ a'x + b'x + c'z &= 0 \end{aligned}\right\} \text{ if } u \neq v \Rightarrow (a,b,c) \neq \lambda(a', b', c')$$

solution space has dimension 1, $q$ solutions, $q - 1$ nonzero solutions

$\#$ of common neighbors: $\frac{q-1}{q-1} = 1$

$$P_{ij} = \begin{cases} v_i v_j : \text{edge if adjacent} \\ v_i z v_j : z \text{ is the unique common neighbor of } v_i v_j \end{cases}$$

$load(v_i) \leq \#$(ordered pairs of neighbors) $\leq (q+1) \cdot q = q^2 + q = n - 1$

⟹ $\xi(G) = n - 1$

Combi # 9

Page 3.

<u>Turán-type problems</u>:

"Forbidden graph" $F$

<u>Def</u>: Turán number: $ex(n,F)$ = max # of edges in a graph on $n$ vertices not containing $F$ as subgraph

e.g: $ex(n, C_4) \sim \frac{1}{2} n^{3/2}$ → for lower bound the same graph works (any two vertices have only one common neighbor)

<u>Theorem</u>: If $\chi(F) = k > 2$ then $ex(n,F) \sim \frac{k-2}{2(k-1)} n^2$

<u>Theorem</u>: Turán's theorem: If $F = K_p$, then the unique largest $F$-free graph is

$p-1$ "equal" classes ( $\lfloor \frac{n}{p-1} \rfloor$ or $\lceil \frac{n}{p-1} \rceil$ ) and draw an edge between two vertices if and only if they are in distinct classes

many edges → large $\omega$

few edges → large $\alpha$

<u>Theorem</u>: For every graph $G$, $\alpha(G) \geq \sum_{v} \frac{1}{d(v)+1}$ ← degree of $v$

<u>1st proof</u>: Construct independent set $S$ by choosing vertex of smallest degree deleting its neighbors

$\alpha(G) \geq sum(G)$

$\alpha(G) \geq 1 + sum(G - v - \text{neighbors})$

$sum(G) = \sum_{v} \frac{1}{d(v)+1}$

neighbors $d(u) \geq d(v)$

$\frac{1}{d(u)+1} \leq \frac{1}{d(v)+1}$

$\alpha(G) \geq 1 + \underbrace{sum(G - v - \text{neighbors})}_{sum(G) - 1}$

2nd proof: delete vertex of largest degree


loss: $\frac{1}{d(v)+1}$

gain: $\sum_u \left( \frac{1}{d(u)} - \frac{1}{d(u)+1} \right)$

$$\alpha(G) \geq \alpha(G-v) \geq \text{sum}(G-v) \geq d(v) \cdot \frac{1}{d(v) \cdot (d(v)+1)} \geq \text{sum}(G)$$

3rd proof: probabilistic


Take random order

Put a vertex $v_i$ into $S$ if it is not preceded by any neighbors

$$\Pr(v_i \in S) = \frac{1}{d(v_i)+1}$$

Claim: $S$ is independent

$$E(|S|) = \sum_{i=1}^{n} \frac{1}{d(v_i)+1}$$

↑ expected value

$$\max = \alpha(G) \geq \mathbb{E}(|S|)$$

Scanned by CamScanner