Biomedical Signal Processing
2018-2019 Autumn

Wavelets II

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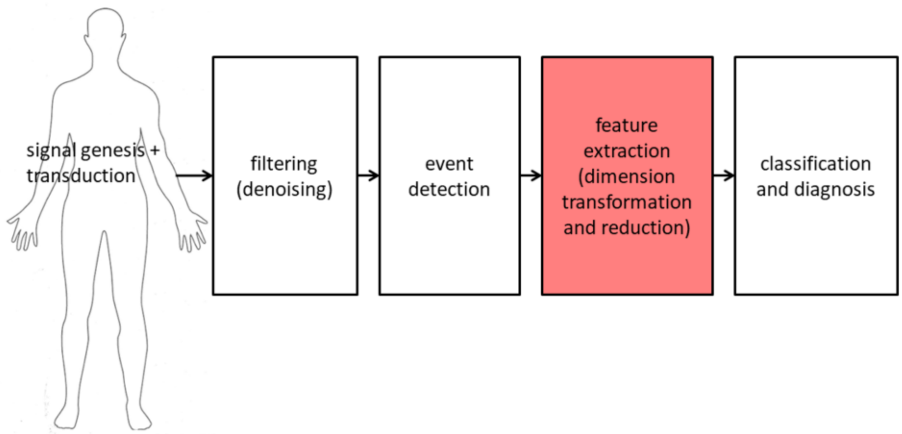
The BSP Flow Chart



Wavelets and Sparsity Trilogy

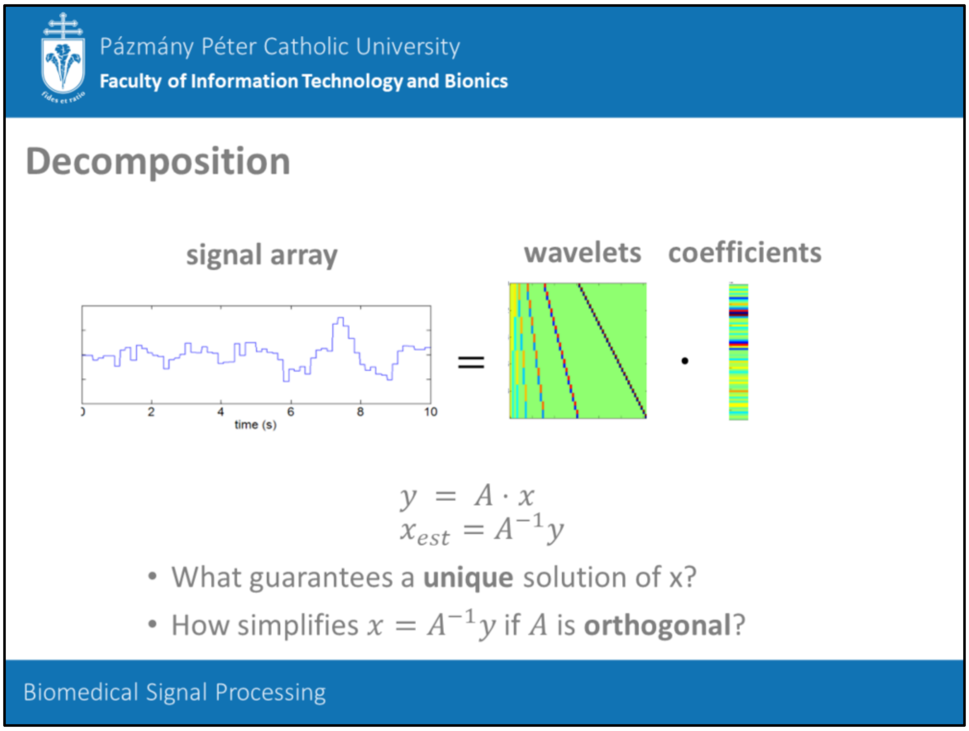
Nov. 05.: Wavelets I: Time-Frequency Representation

Nov. 12.: Wavelets II: Decomposition

Nov. 19.: Sparsity

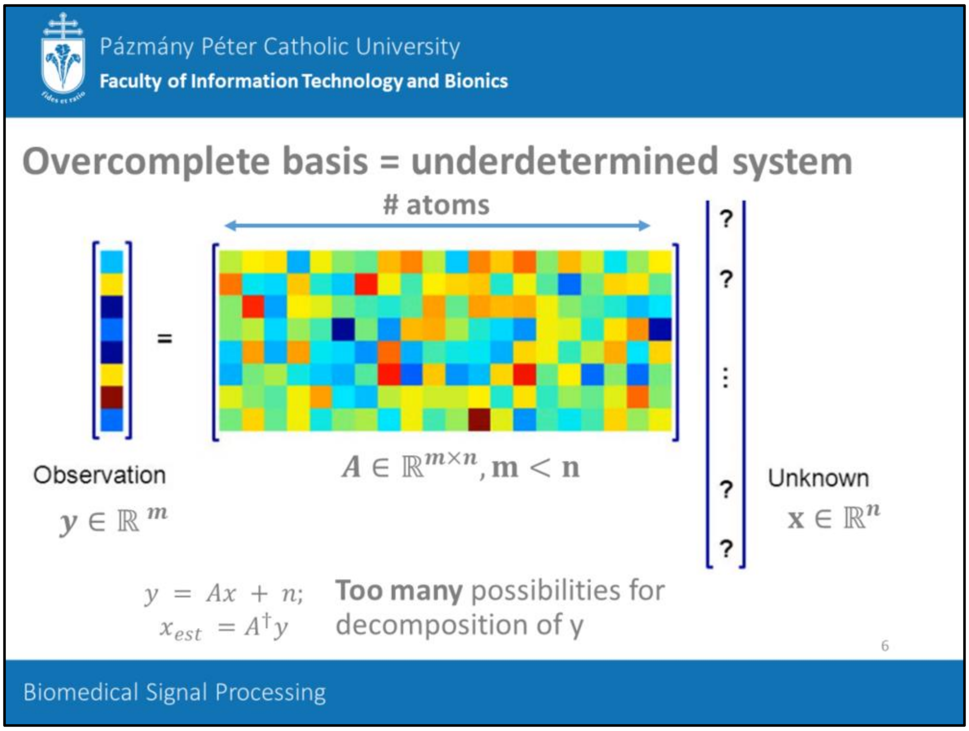
Today's goal

- How can a signal be decomposed using wavelets? How does it depend on the completeness and orthogonality of the wavelet basis?
- From CWT to DWT – what is the difference?
- Multiresolution analysis: filtering and compressing using the wavelet coefficients
- Wavelet properties



We would like to have a wavelet basis, where any signal can be decomposed uniquely.
Remember from linear algebra:
Rank of matrix **A should equal dimension of x** for **unique** solution
according to https://en.wikipedia.org/wiki/Rank_(linear_algebra):
rank is "dimension of the vector space generated (or spanned) by its columns";
"maximal number of linearly independent columns of *A*".
The row and column ranks are the same!

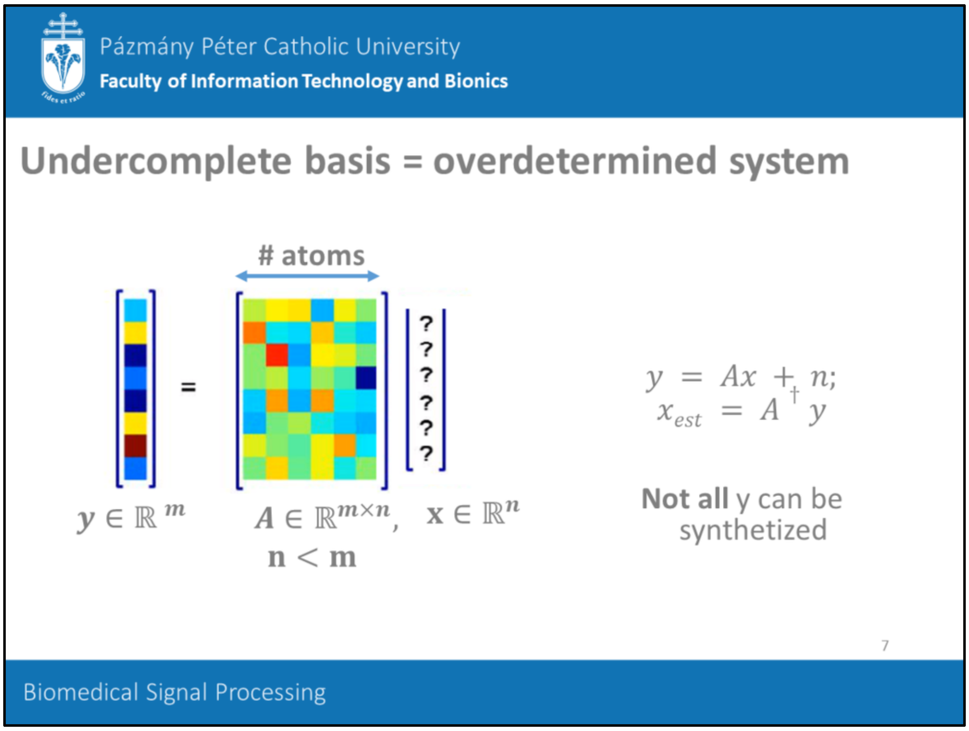
with orthogonal (or more precisely orthonormal) decomposition, $A^T = A^{-1}$, so that x =$A^T y$. (transpose of matrix equals its inverse). This is computationally more efficient



•projection: many different x's could have produced y
•not given enough constraints (undetermined system): can provide any number of solutions
•too many atoms with which to solve the problem: overcomplete basis x

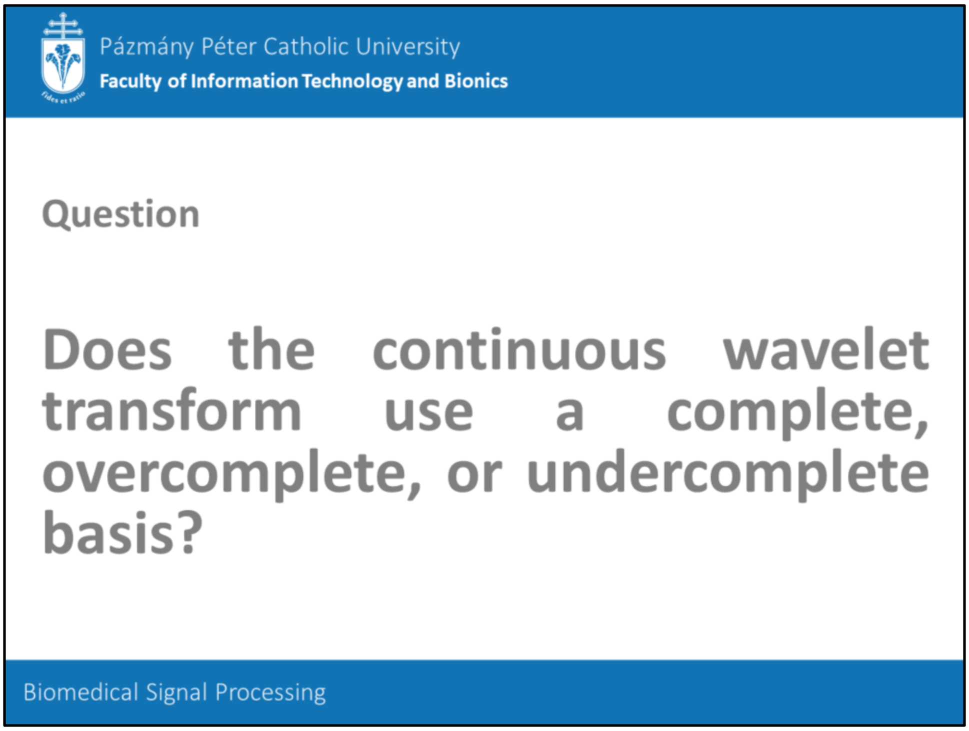
•For obtaining x the inverse of A has to be calculated. However, A is not square, meaning that the regular formula is not worknig. Instead this, the Moore-Penrose pseudoinverse is calculated:
$A^{\dagger} = pinv(A) = (AA^T)^{-1}A^T;$

% minimise $\|x_{est}\|_2$ subject to y = Ax_{est}
Here the l2 minimization of x is a constrain, helping to choose one solution from the infinite set of solutions

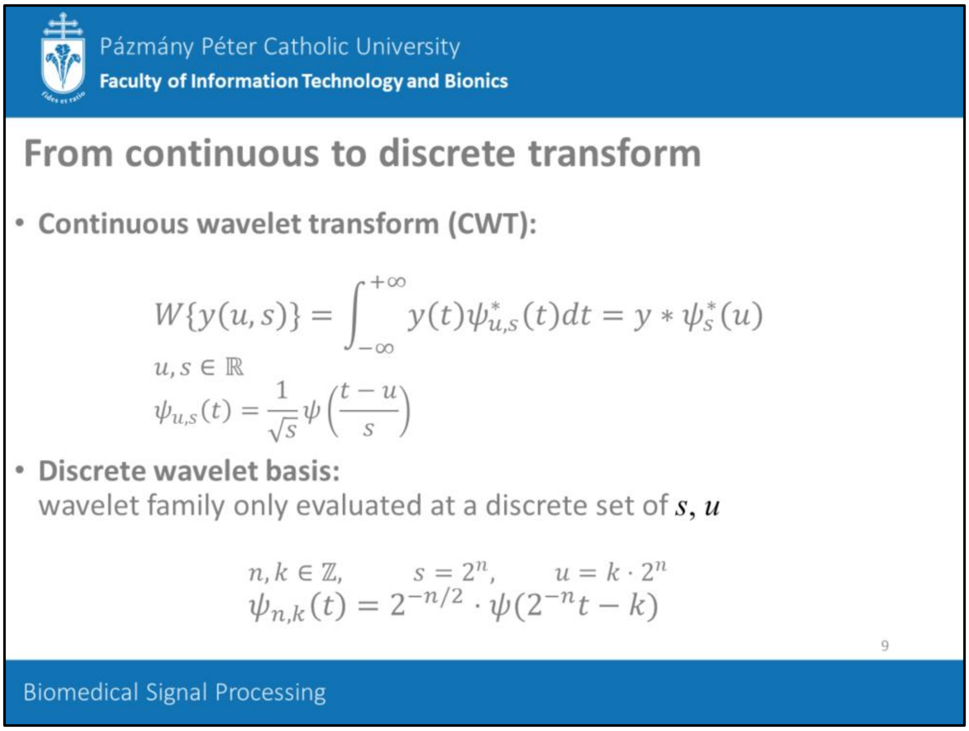


not all y's can be synthesised using just a few atoms ($\dim(x) < (\dim(y))$: we are asking for too much (overdetermined system with too many constrain not allowing a solution) that we cannot synthesise with our undercomplete basis (set of atoms)

$A^{\dagger} = pinv(A) = (AA^T)^{-1}A^T;$



Overcomplete, see last lecture – windowed fourier's overcompleteness-explanation



ψ is the motherwavelet

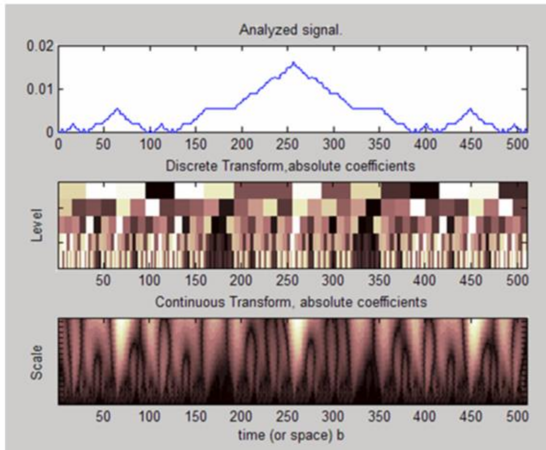
For the CWT if we want to get the spectrogram at (u,s), we calculate the similarity of the signal to the wavelet with (u,s).
This is the same, as taking the convolution of the signal with the wavelet at scale s (and zero shift, u, as the convolution will take all possible shifts), and taking the u^{th} value from the resulting array. So the convolution gives a whole row of the spectroram.

The power of 2 (dyadic) in the discrete u and s will guarantee the orthogonality of the atoms → completeness of the system.

(Orthogonal arrays are independent, maximum number of them is the dimension → complete system with unique solution)

Mallat: A wavelet tour of signal processing pp. 3,92,102

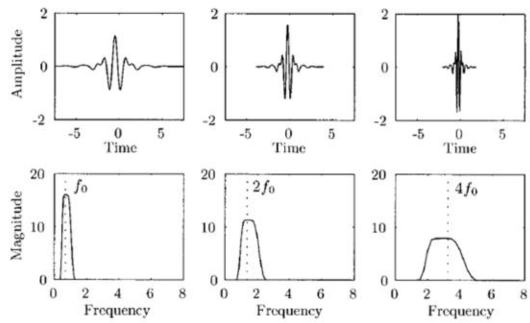
From continuous to discrete



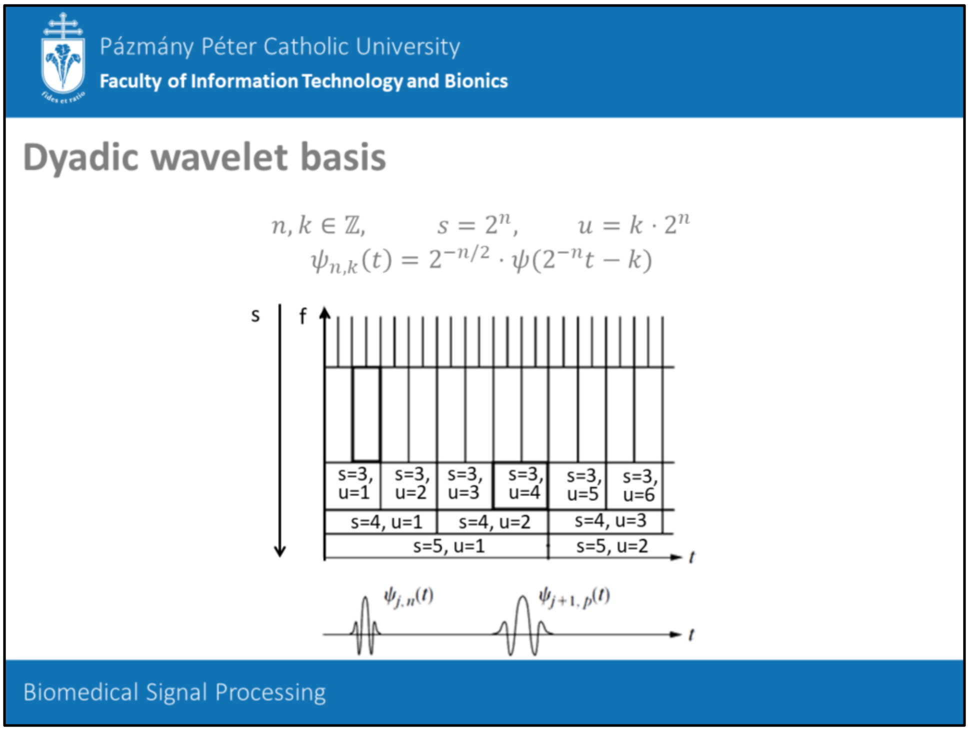
https://it.mathworks.com/help/wavelet/ref/cwtold.html

Dyadic decomposition

In the choice of wavelets with $n, k \in \mathbb{Z}$, $s = 2^n$, $u = k \cdot 2^n$, 2^n guarantees a complete and non-redundant basis



Sörnmo and Laguna: Bioelectrical Signal Processing in Cardiac and Neurological Applications, p. 289



Remember the uncertainity boxes:

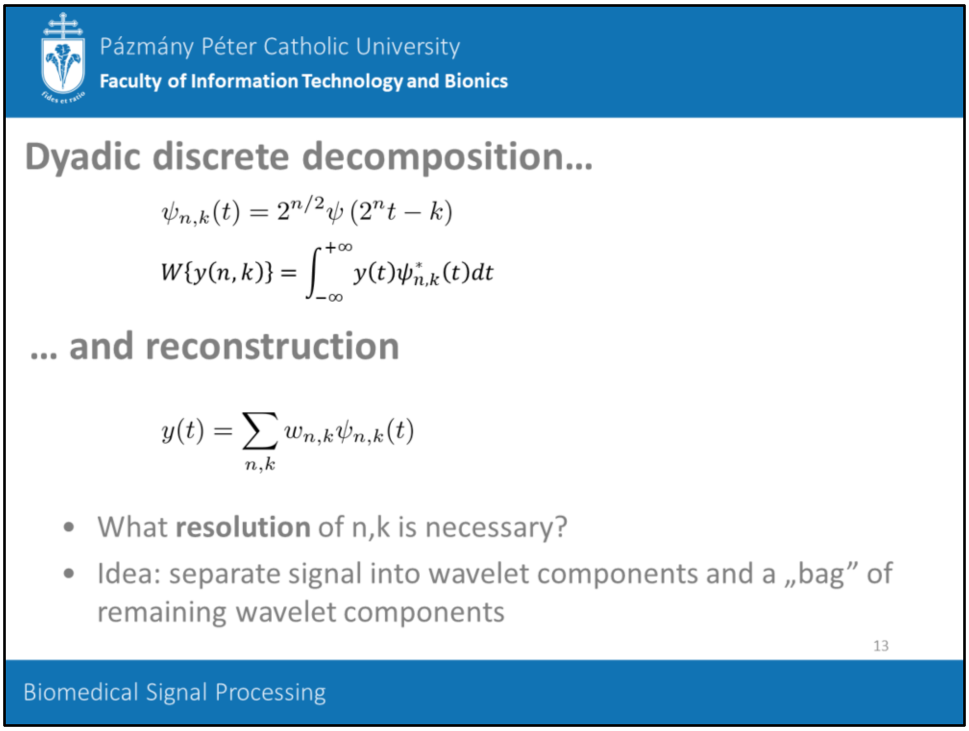
Short wavelets have bad frequency resolution, good time resolution (on the top tall but small-width rectangles)

Long wavelets have good frequency resolution, bad time resolution (in the bottom, wide but small-height rectangles)

With increasing s, the function $\psi\left(\frac{t-u}{s}\right)$ will be more dilated, „streched out” → lower frequency, longer timewindow

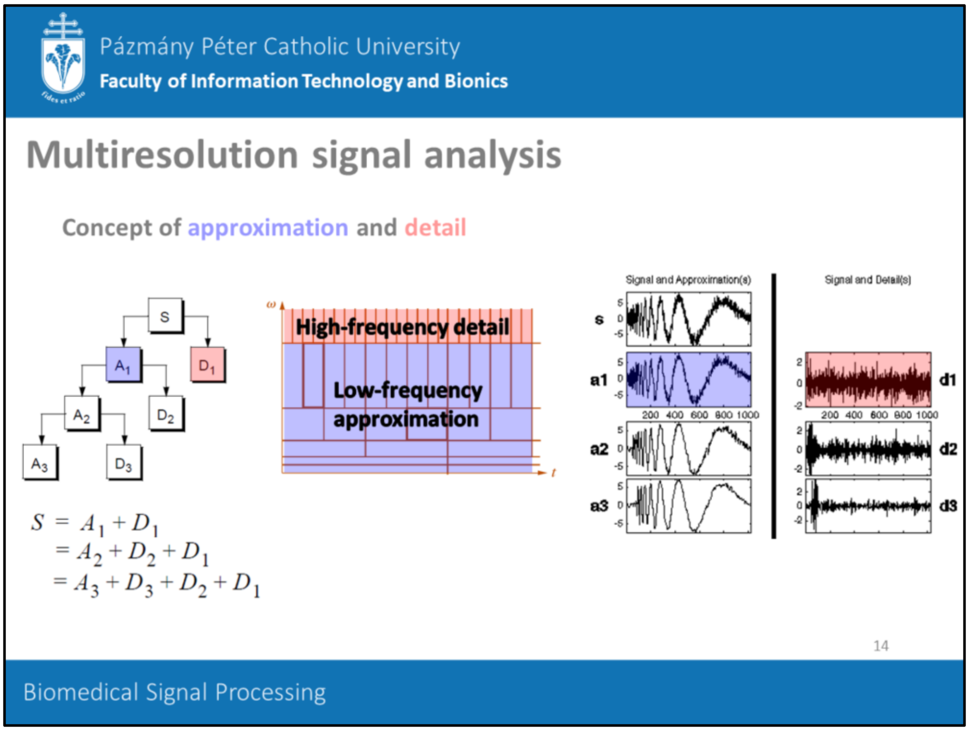
Mallat: A wavelet tour of signal processing, pp. 20

Sörnmo and Laguna: Bioelectrical Signal Processing in Cardiac and Neurological Applications, pp. 290-291



If not only the baisis, but the signal too is discrete, $w_{n,k}(t) = \int_{-\infty}^{+\infty} y(t)\psi_{u,s}^*(t)dt$ turns into a sum too.

Small coefficients most likely represent noise. This way we can efficiently filter our signal: we discard the small-amplitude compinents.



Approximate coefficient is a smoothed version of the original signal, while the detail coefficient is the high frequencies included into the original signal. Approximate coefficients can also be decomposed into another approximate and detail coefficients.

Mallat: A wavelet tour of signal processing, p. 20; Matlab Wavelet Toolbox User's Guide v1, 1-19, 1-27, 3-18;
Sörnmo and Laguna: Bioelectrical Signal Processing in Cardiac and Neurological Applications, pp. 292-297

for added flexibility/confusion, look up wavelet packets...

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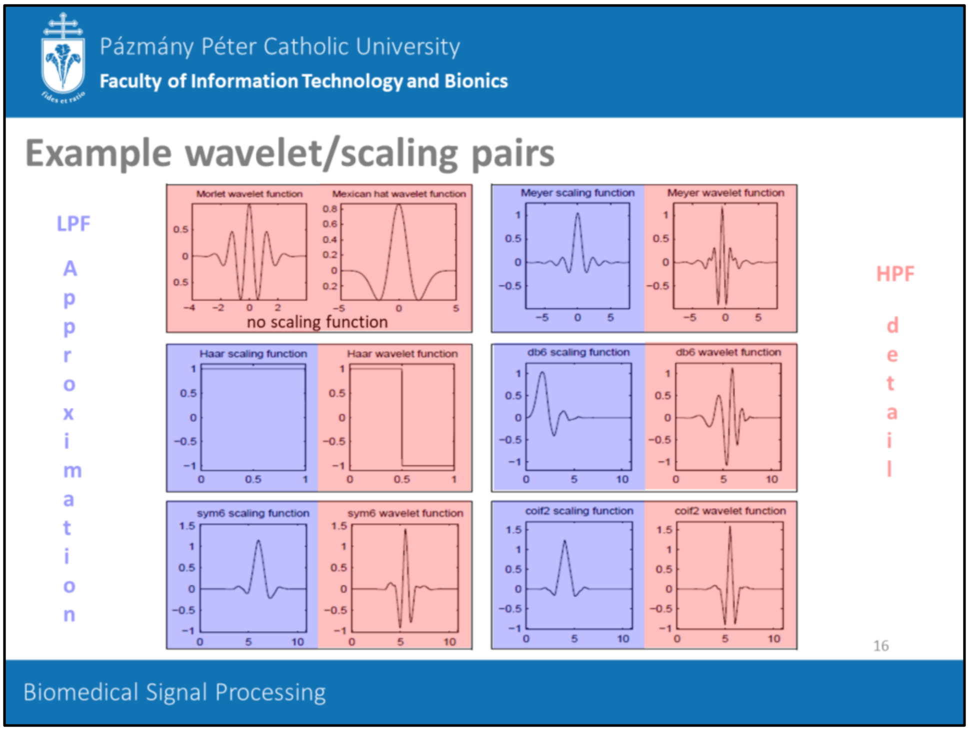
Multiresolution signal analysis

- Approximation and detail achieved using low-pass filter f and high-pass filter g
- HPF g: discretised wavelet ψ at lowest scale
- LPF f: discretised scaling function Φ
- What is Φ? Any function that satisfies:
  - orthonormality between discrete translations of Φ at same scale
  - orthonormality between Φ and ψ at same scale

Remember, that low scale == high frequency, short time window



integral of wavelet ψ = 0
integral of scaling Φ = 1

- wavelet and scaling complement each other,
- ψ+Φ covers entire spectrum of interest – they will split the signal into 2
- however, phase of Φ not determined by this requirement

Matlab Wavelet Toolbox User’s Guide v1, 6-7

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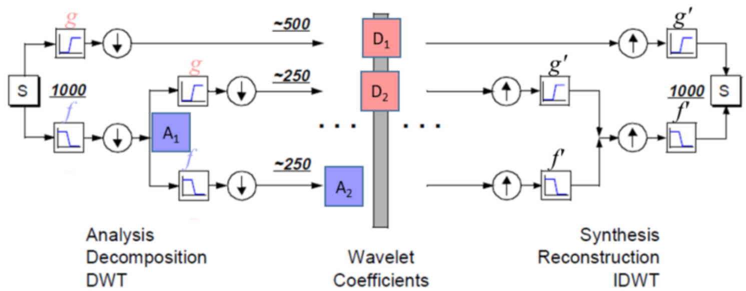
Fast discrete decomposition/reconstruction



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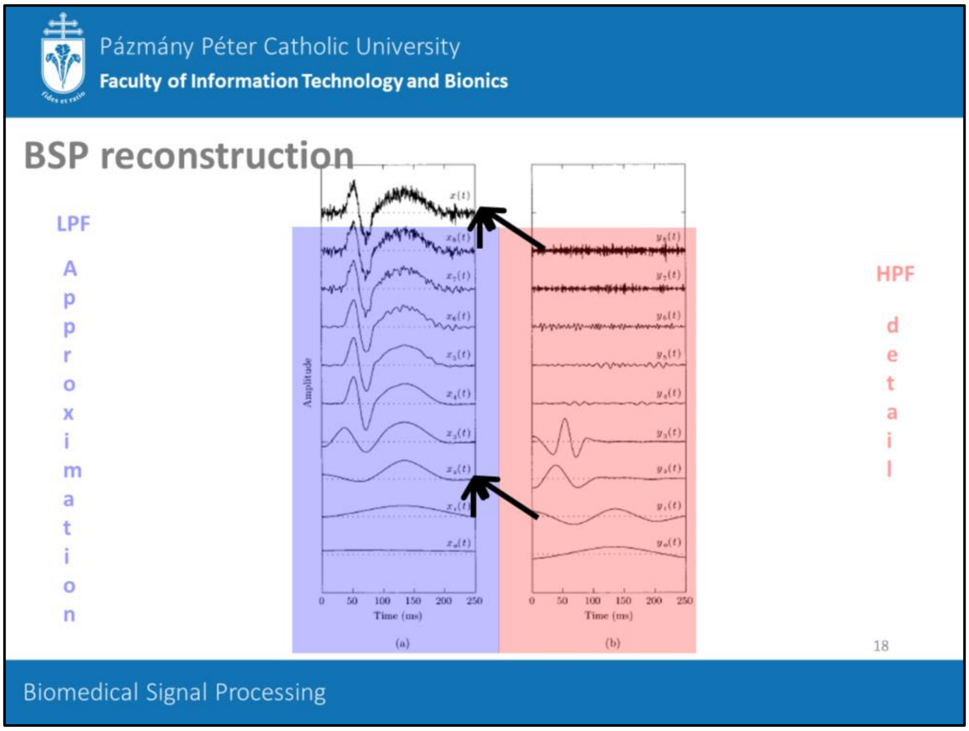
Biomedical Signal Processing

Given a signal s of length N, the DWT consists of at most $\log_2 N$ steps. Starting from *s*, the first step produces two sets of coefficients: approximation coefficients A_1 and detail coefficients D_1. Convolving *s* with the decomposition lowpass filter f and the decomposition highpass filter g (defined by the wavelet family,), followed by dyadic decimation (downsampling), results in the approximation and detail coefficients respectively (the convolution by the filter is: $y * \psi_s^*(u)$). The next step splits the approximation coefficients A_1 in two parts using the same scheme, replacing *s* by A_1, and producing A_2 and D_2, and so on.

Decomposition of signal S to required level by recursive filtering f, g and dyadic $\downarrow\times 2$ downsampling (to obtain the next scale, and the same filter can be used)
Approximation and detail recovery: $\uparrow\times 2$ (insert 0 between samples) and apply conjugate mirror filters f', g'
NB: Existence of f', g' non-trivial result due to $\downarrow$ aliasing

iterative $\uparrow\times 2$ and convolution of f' with itself leads to Φ;
iterative $\uparrow\times 2$ and convolution of g' with itself leads to ψ

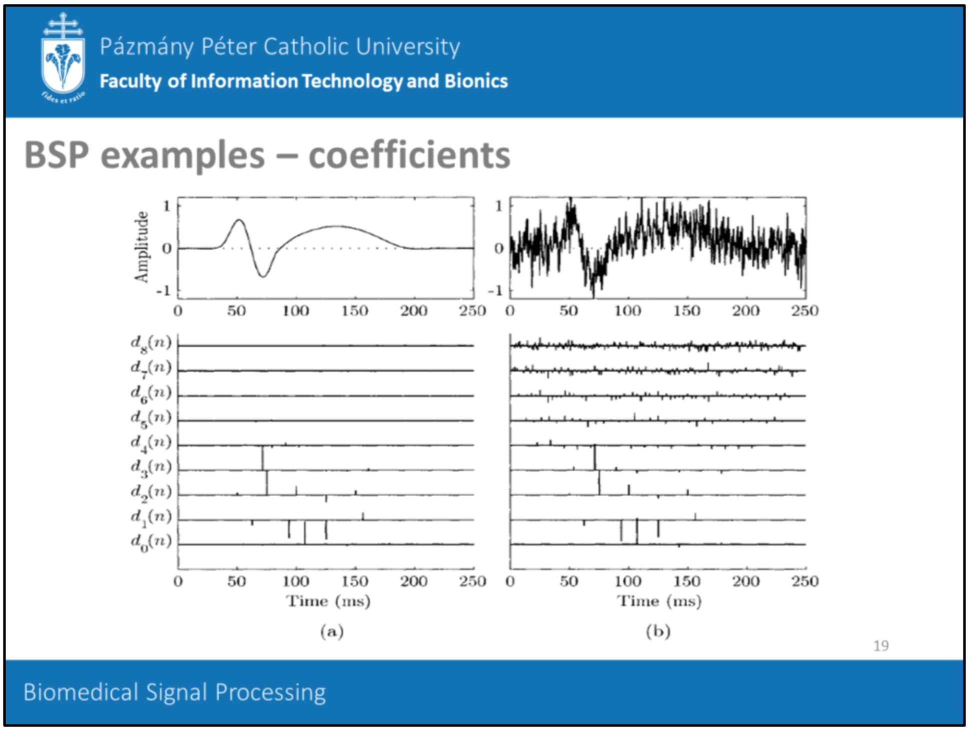
Matlab Wavelet Toolbox User's Guide v1, 1-23 – 1-25



Multiresolution analysis of evoked potential using the Coiflet-4.
(a) The approximation signals (b) the detail signals
at different scales. The original signal is shown at the top left of the figure.

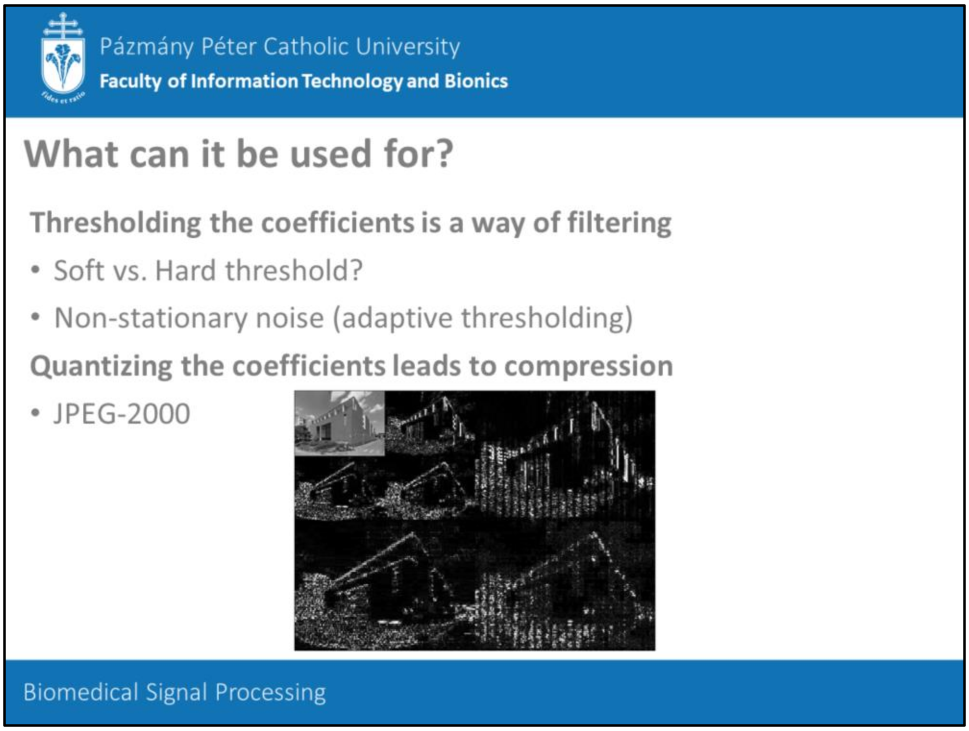
If noise filtering is needed, the reconstruction stops after some step – high resolution details are not added back.
Or, low amplitude coefficients are discarded → also filtering

Sörnmo and Laguna: Bioelectrical Signal Processing in Cardiac and Neurological Applications, pp. 311-312



(a) An evoked potential and the related DWT using the Daubechies-4 wavelet.
(b) The same signal as in (a), but with noise added.
Note that the noise is concentrated in the detail coefficients of the finest scales

Sörnmo and Laguna: Bioelectrical Signal Processing in Cardiac and Neurological Applications, pp. 311-312



JPEG:
Wavelet transform:
irreversible: the CDF 9/7 wavelet transform. It is said to be "irreversible" because it introduces quantization noise that depends on the precision of the decoder.
After the wavelet transform, the coefficients are scalar-quantized to reduce the number of bits to represent them, at the expense of quality. The output is a set of integer numbers which have to be encoded bit-by-bit. The parameter that can be changed to set the final quality is the quantization step: the greater the step, the greater is the compression and the loss of quality.

quantization
https://en.wikipedia.org/wiki/JPEG_2000#Quantization

denoising, soft or hard thresholds:

https://www.mathworks.com/help/wavelet/ug/wavelet-denoising-and-nonparametric-function-estimation.html#f8-22146

denoising, adaptive tresholding:
https://www.mathworks.com/help/wavelet/ug/wavelet-denoising-and-nonparametric-function-estimation.html#f8-50819

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Property	Explanation
Analytic *(no negative frequencies)*	•result: real and imaginary part 90° out of phase with each other •allows reconstruction
Orthogonal *(wavelets are orthogonal)*	•makes orthogonal decomposition possible •preserves energy
Biorthogonal *(invertible, but not necessarily orthogonal)*	•many vanishing moment •linear phase (important for images)
Compact support *(zero outside some boundary)*	•numerically simple (often good approx. for non-compact bases) •length influences the recoverable feature-size
Existence of scaling function	•necessary for fast orthogonal decomposition
Explicit	•can wavelet be defined by analytical expression or are functional equations necessary to express them?
Number of vanishing moments *More moments, sparser representation possible*	•remove mean, DC, parabolic, etc. from signal •useful for compression
Regularity *how many times differentiable?*	•regular wavelets () extract smooth signals •irregular wavelets () highlight fractal structure
Symmetric	•linear phase (important in e.g. image processing/compression)

Biomedical Signal Processing

Matlab Wavelet Toolbox User's Guide v1, 6-62

The type of wavelet analysis best suited for your work depends on what you want to do with the data.

If your goal is to perform a **detailed time-frequency analysis**, choose the continuous wavelet transform (**CWT**).

- The CWT is superior to the short-time Fourier transform (STFT) for signals in which the instantaneous frequency grows rapidly, such as in a hyperbolic chirp.
- The CWT is good at localizing transients in nonstationary signals
- **Analytic** wavelets are used for cwt (Morse, Morlet, Bump). These wavelets have one-sided spectra, and are complex valued in the time domain (we have seen such thing with Hilbert-transform). Because the wavelet coefficients are complex valued, the CWT provides **phase information: it means that** signal can be **reconstruct**ed, as both magnitude and phase information is needed for reconstruction.

In a **multiresolution analysis** (MRA), you approximate a signal at progressively **coarser scales** while recording the **differences between approximations** at consecutive scales. You create the approximations and the differences by taking the **discrete wavelet transform** (DWT) of the signal.

- The DWT provides a **sparse representation** for many natural signals.
- On a $\log_2$ scale, the difference between consecutive scales is always 1. In the case of the CWT, differences between consecutive scales are finer.

- When generating the MRA, you can either subsample (decimate) the approximation by a factor of 2 every time you increase the scale or not. In the **decimated** DWT, translations are integer multiples of scale (leads to **minimally redundant ~ orthogonal**). For the **nondecimated** DWT, translations are integer shifts. A nondecimated DWT provides a **redundant** representation of the original data, but not as redundant as the CWT. Your application not only influences your choice of wavelet, but also which version of the DWT to use.

If preserving energy in the analysis stage is important, you must use an orthogonal wavelet. An **orthogonal transform preserves energy**. An orthogonal basis also makes the calculation of the coefficients easier: $x = A^T y$. Consider using an orthogonal wavelet with compact support. Keep in mind that except for the Haar wavelet, orthogonal wavelets with compact support are **not symmetric** – they have nonlinear phase, making exact reconstruction impossible.

If you want to find **closely spaced features,** choose **wavelets with smaller support**, such as haar, db2, or sym2.

Minimally redundant representations obtained with decimated **DWT**, using orthonormal wavelet.families are a **good choice for compression**, when you want to remove features that are not perceived. An **orthogonal** wavelet, such as a Symlet or Daubechies wavelet, is a good choice for **denoising** signals.
A **biorthogonal** wavelet can also be good for image processing (but not energy-preserving). Biorthogonal wavelet filters are symmetric, having **linear phase** which is a very critical for image processing. Using a biorthogonal wavelet filter will not introduce visual distortions in the image. Using a wavelet with **many vanishing moments** (like biorthogonal wavelets) results in fewer significant wavelet coefficients → compression is improved. They can also have better regularity (n-times differentiable) → smoother reconstruction.

The CWT of a signal provides a highly redundant representation of a signal. There is significant overlap between wavelets within and across scales. Also, given the fine discretization of the scales, the **cost to compute the CWT** and store the wavelet coefficients is significantly **greater than the DWT**.

https://de.mathworks.com/help/wavelet/gs/choose-a-wavelet.html

Useful references

- Matlab Wavelet Toolbox https://www.mathworks.com/products/wavelet.html

help wavelet

waveletAnalytzer (also reached by Apps->WaveletAnalyzer)

waveletSignalDenoiser (ditto)

Student suggestion:

- Wavelet tutorial http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html