Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

Biomedical Signal Processing
2018-2019 Autumn

Wavelets I

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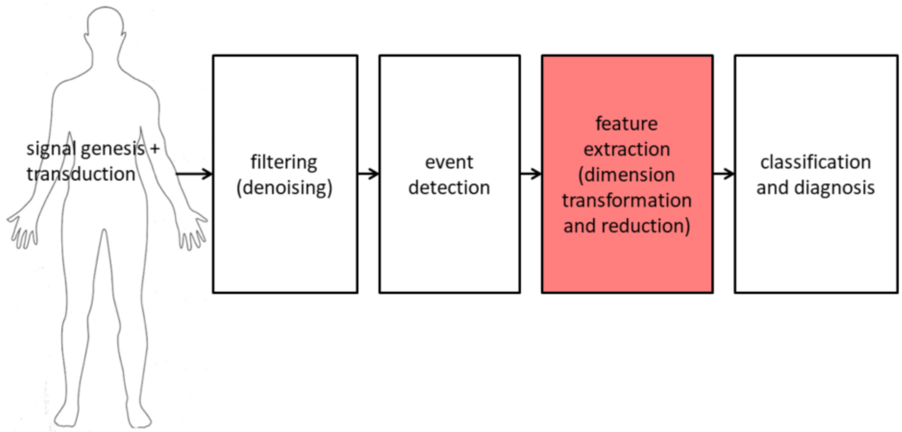
Responsible lecturer: dr. Miklós Gyöngy

The BSP Flow Chart



Wavelets and Sparsity Trilogy

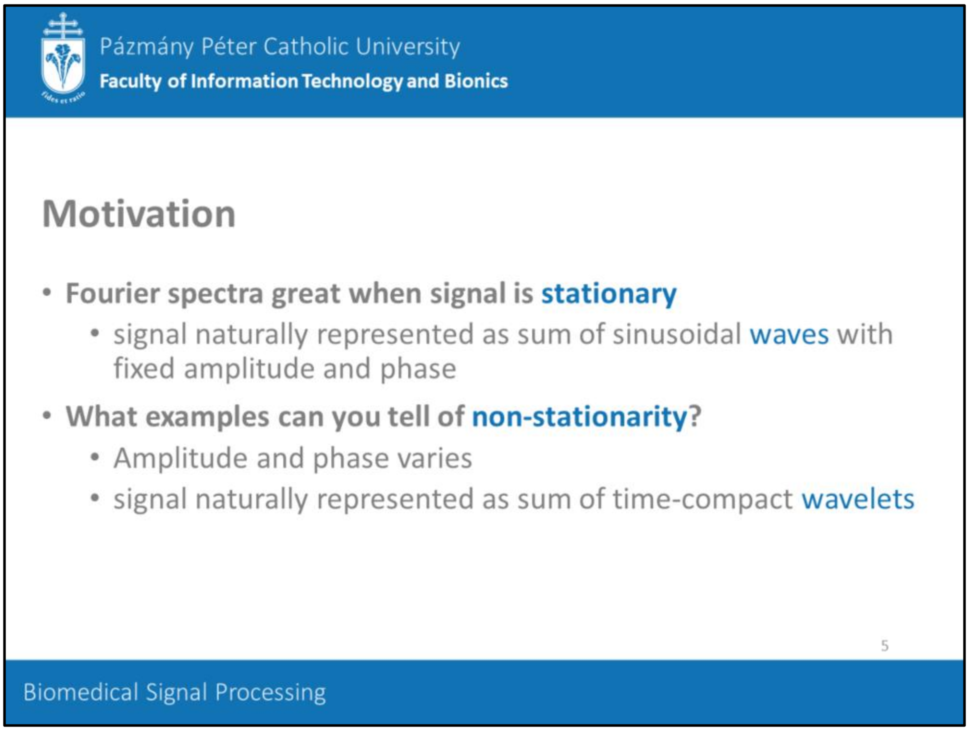
Nov. 05.: Wavelets I: Time-Frequency Representation

Nov. 12.: Wavelets II: Decomposition

Nov. 19.: Sparsity

Today's goal

- How to represent the **frequency changes** in a signal? Understanding the **spectrogram**, and its calculation methods.
- Following the **Heisenberg uncertainity**, deriving the limit of the frequency-time-resolution.
- From the **Gabor-wavelets** to the C**ontinuous Wavelet Transform**.



When the signal is stationary, it can be composed from constant-amplitude sinusoidal waves. (The coefficients c_ks are not time-dependent)

The non-stationarity of a signal can be caractherized many different ways:

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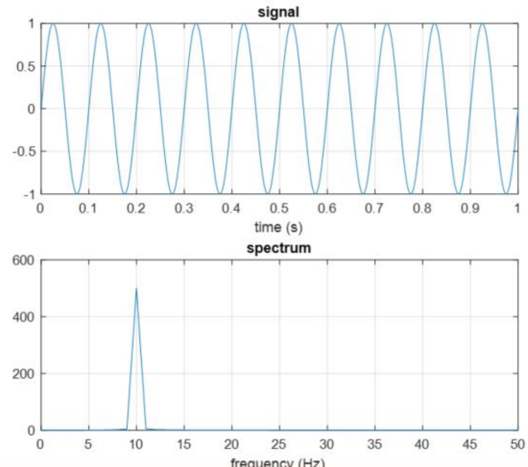
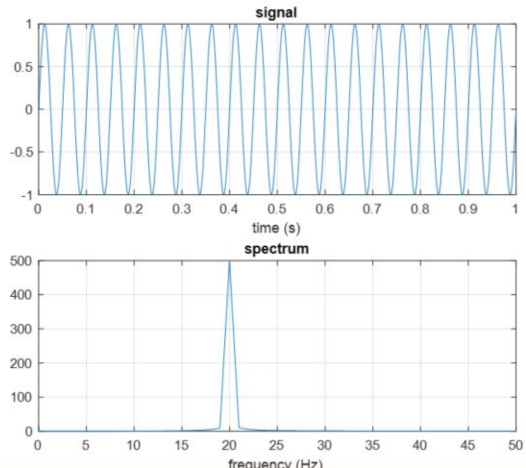
Fourier Examples



Fourier Examples



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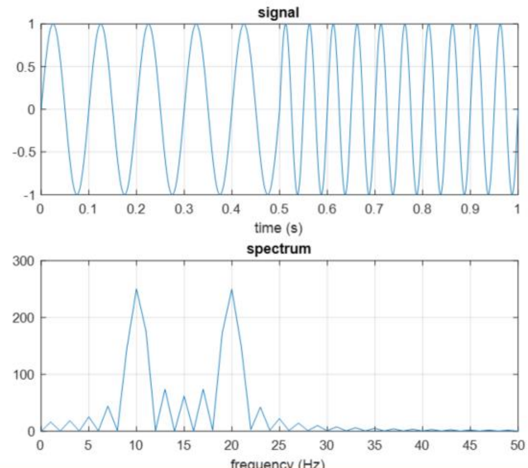
Fourier Examples



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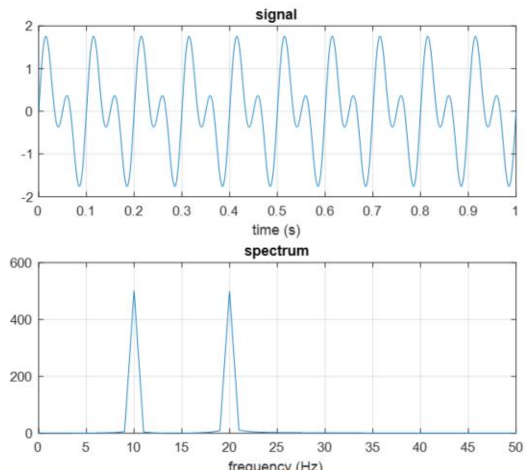
Two frequencies are present in the signal, but not simultaneously. From the spectrum we can not tell, whether they are separated in time, or happen together (next slide)

Fourier Examples



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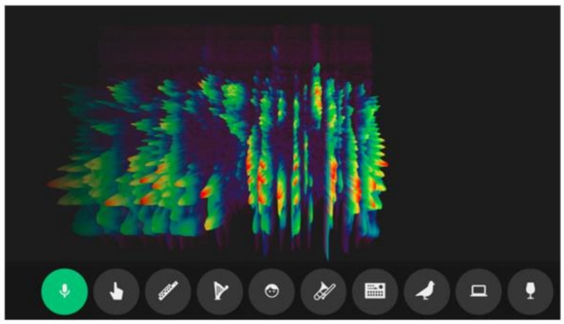
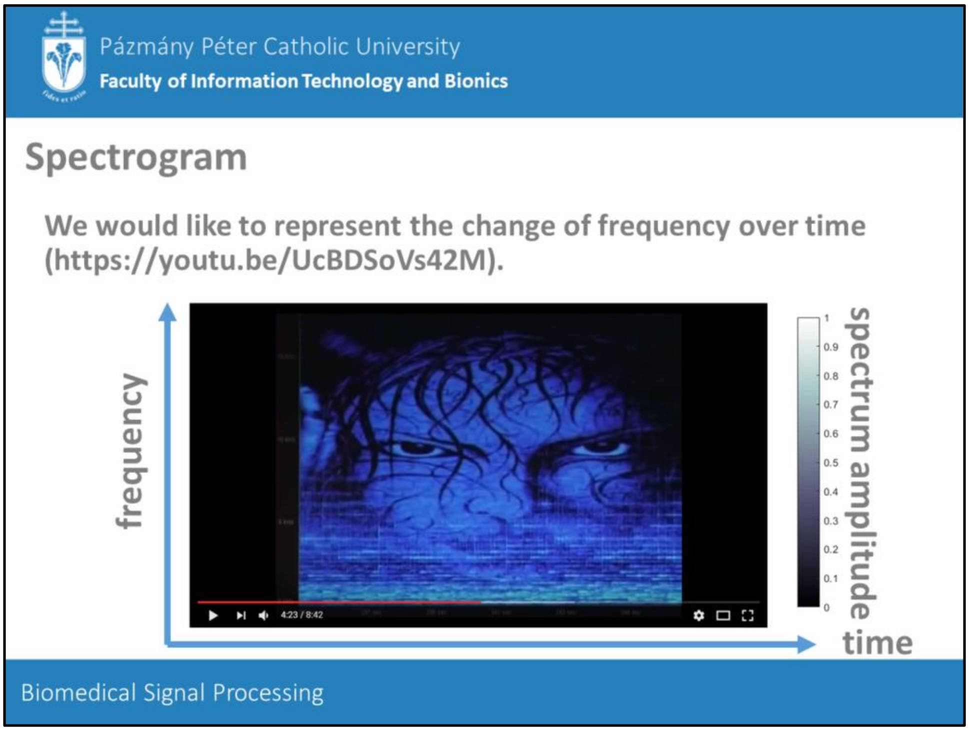
Spectrogram

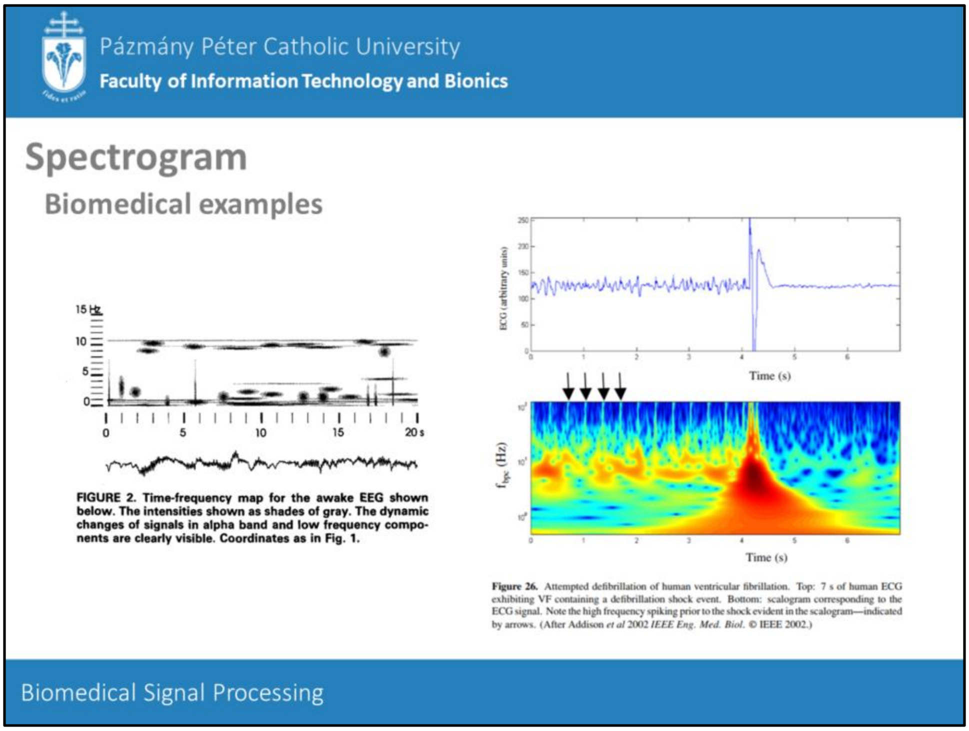
We would like to represent the change of frequency over time (https://musiclab.chromeexperiments.com/Spectrogram/).





We would like to have a plot, where the frequency components corresponding to a time instant are represented with a colorbar.

In this kind of ‚music' an image is embedded in the spectrogram. What does it sound like?



Biomedical examples:
The change of **EEG oscillation** frequencies in an awake subject (the amplitude of the frequency component is plotted in gray dots: the bigger/darker the dot, the stronger the frequency)
The spectrogram of a **fibrillation** on an **ECG**. From the tim-frequency representation it is easy to read the change.

http://doi.org.ololo.sci-hub.cc/10.10
https://www.ncbi.nlm.nih.gov/pubmed/750346288/0967-3334/26/5/R01

SOLUTION?

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Windowed Fourier transform

- **normal Fourier:**

$$Y(f) = F\{y(t)\} = \int_{-\infty}^{\infty} y(t)\ \exp(-j2\pi ft)\ \mathrm{d}t$$

- **windowed Fourier:**

$$\begin{aligned} Y(f,u) &= F\{a(t-u)y(t)\} \\ &= F\{a(t-u)\} * F\{y(t)\} \\ &= \exp(-j2\pi uf)\ A(f) * Y(f) \end{aligned}$$

This is an overcomplete basis, leading to ambiguity of the frequency

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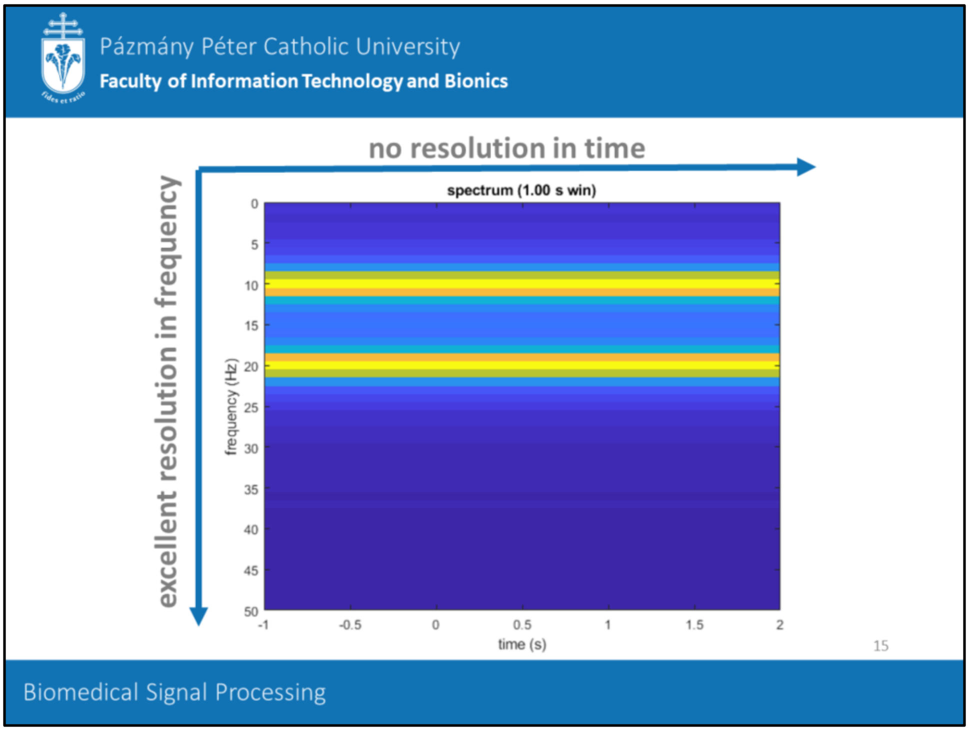
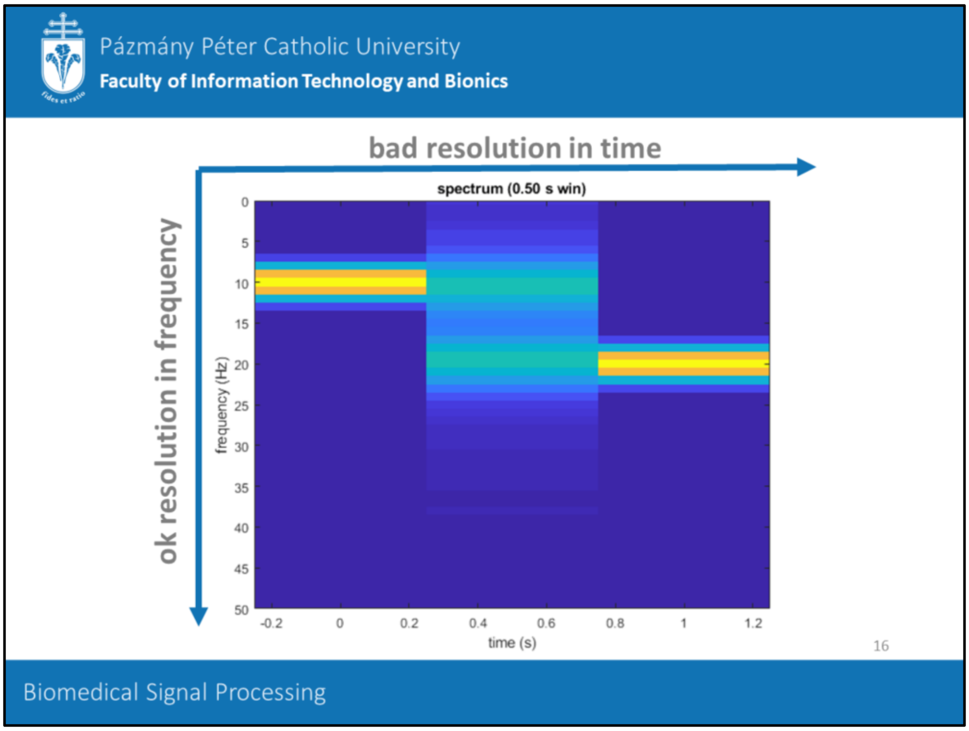
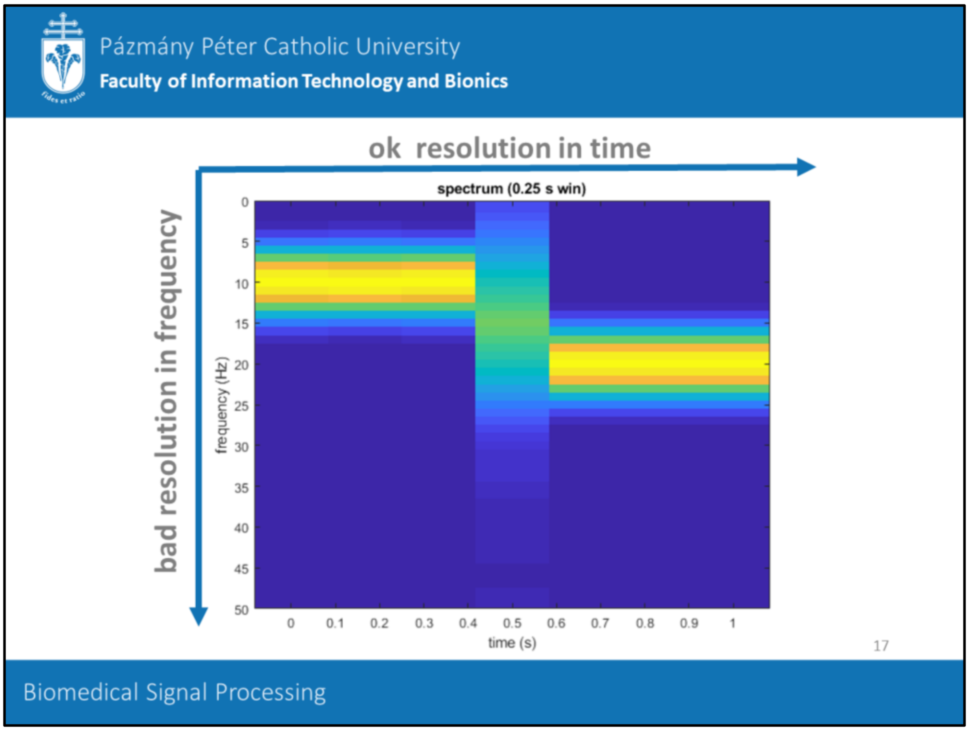
why would we want to do that?
does windowed Fourier transform create an undercomplete or overcomplete basis?
overcomplete, because several instances of Y(f,u) can produce the same signal (e.g. modulate sine);
so the transform will probably contain another implicit constraint in it (such as miminum l2 energy)
this overcomplete basis, or underdetermined system causes an ambiguity in frequency
how can we quantify this ambiguity?



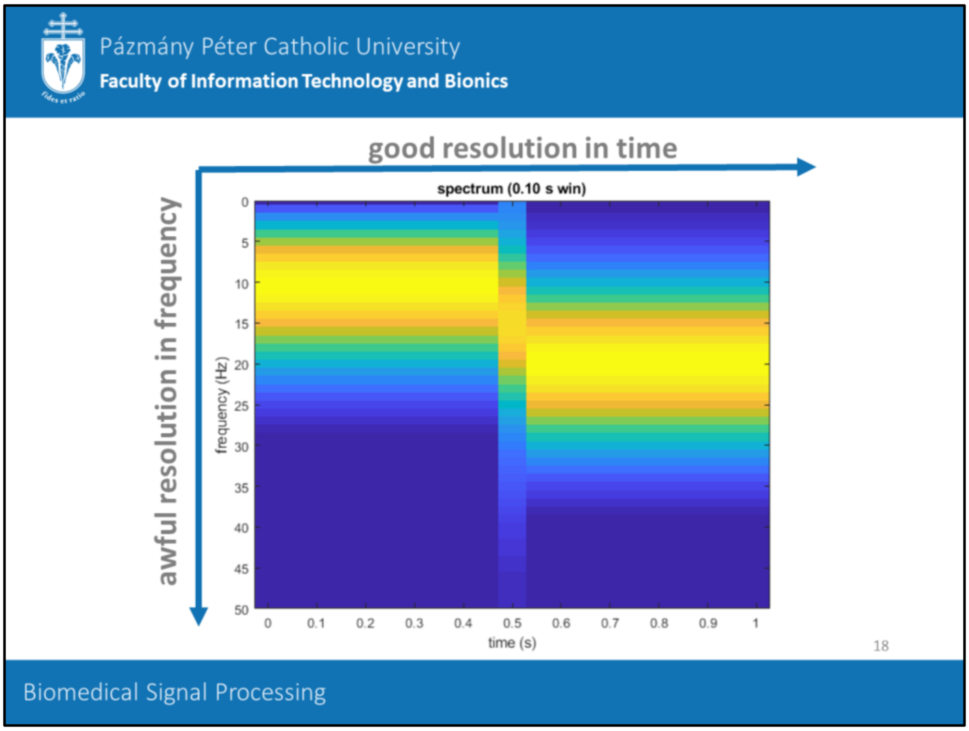
full window: no time resolution



half window (with half overlap), more temporal resolution



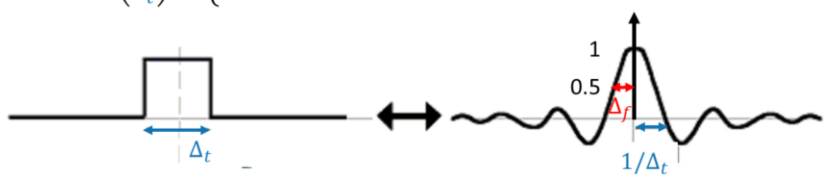
even better temporal resolution, but with larger frequency uncertainty



and even higher frequency uncertainty, but now with good localization of temporal change

Rectangular Fourier window

$$a(t) = \prod\left(\frac{t}{\Delta_t}\right) = \begin{cases} 1 & if\ t < \Delta_t/2 \\ 0 & otherwise \end{cases}$$



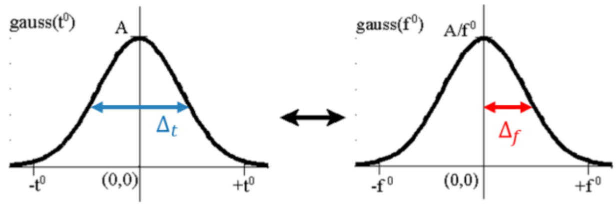
$$A(f) = \frac{\Delta_t(\sin(\pi f \Delta_t))}{\pi f \Delta_t} = \Delta_t \mathrm{sinc}(\Delta_t f)$$

$\mathrm{sinc}(0.603) \approx 0.5$

$\Delta_t \cdot \Delta_f \approx 0.603$

The Fourier transform of a rectangular funtion is a sinc f

Gaussian Fourier window (Gabor)



$a(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2}$

$\mathcal{F}\big(a(t)\big) = e^{-2(\pi f \sigma)^2}$

$FWFM\ (\Delta_t)$:

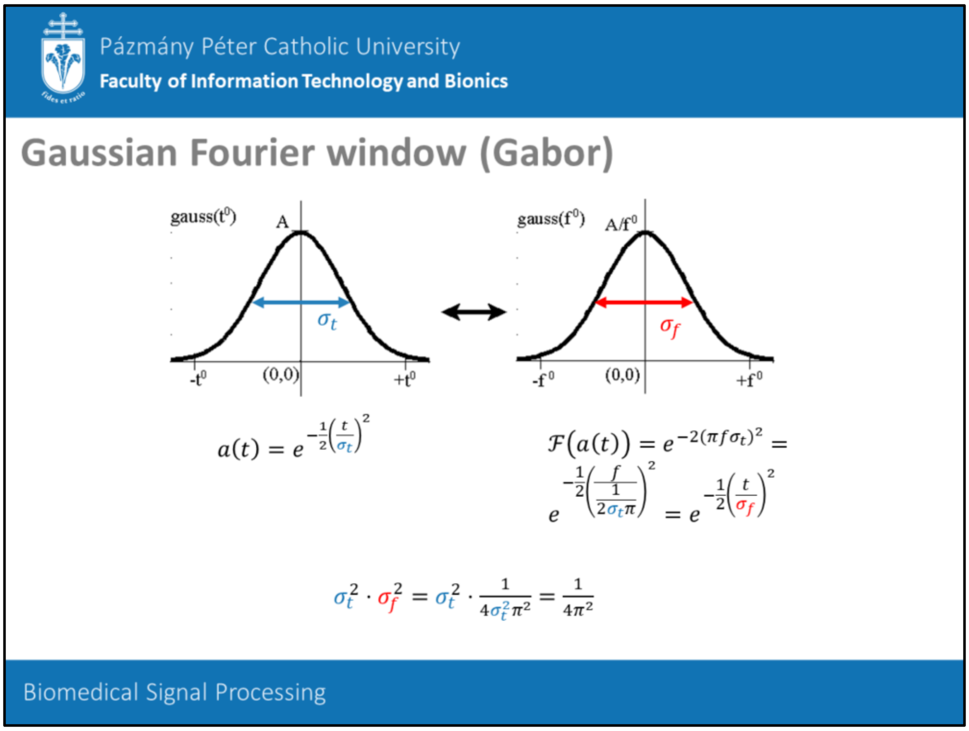
$\Delta_t = \pm 2\sigma\sqrt{2\ln(2)}$

$HWHM(\Delta_f)$:

$\Delta_f = \pm\sqrt{\left(\frac{\ln(2)}{2\pi^2\sigma^2}\right)}$

$$\Delta_t \cdot \Delta_f = \frac{2\ln(2)}{\pi} \approx 0.4412$$

product of two widths much lower in case of Gaussian – better compromise can be reached



product of two widths lower in case of Gaussian – better compromise can be reached for the time-frequency uncertainity:

$FWFM\ (\Delta_t)$:

$$e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2} = \frac{1}{2} = 2^{-1}$$

$$-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2 = -\ln(2)$$

$$\Delta_t = \pm 2\sigma\sqrt{2\ln(2)}$$

$HWHM(\Delta_f)$:

$$e^{-2(\pi f \sigma)^2} = 2^{-1}$$

$$-2(\pi f \sigma)^2 = -\ln(2)$$

$$\Delta_f = \pm\sqrt{\left(\frac{\ln(2)}{2\pi^2\sigma^2}\right)}$$

$$\Delta_t \cdot \Delta_f = \sqrt{\frac{(8 \cdot \ln^2(2)\sigma^2)}{2\pi^2\sigma^2}} = \frac{2\ln(2)}{\pi} = 0.4412$$

This uncertainity can also be expressed by the variance of the two gaussian functions:

T-F Uncertainty Principle

- Product of variance in time and frequency never smaller than $\frac{1}{4\pi^2}$

$$\sigma_t^2\sigma_f^2 \geq \frac{1}{4\pi^2}, \text{ where } \sigma_t^2 = \frac{\int t^2|y(t)|\mathrm{d}t}{||Y(t)||^2},\ \sigma_f^2 = \frac{\int f^2|Y(f)|\mathrm{d}f}{||Y(f)||^2}$$

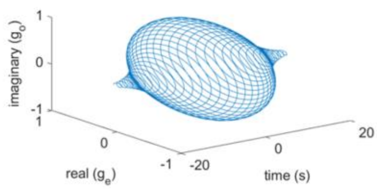
- Equality satisfied in case of Gaussian and Gabor functions (Gaussian modulated sinusoids)

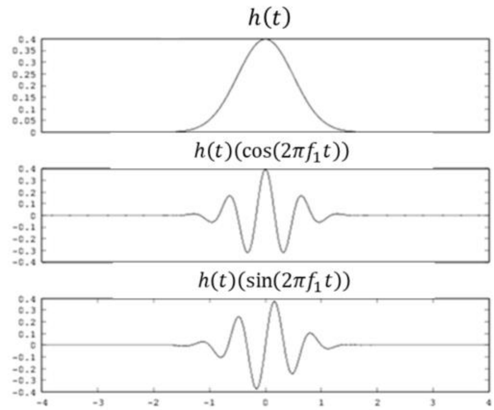
http://www.ee.iitb.ac.in/student/~pawar/Wavelet%20Applications/Chapters_review/ch03_Gr3_Gr2.pdf

Gabor function

$$g(t) = g_e(t) + i g_o(t) = h(t)\left(\cos(2\pi f_1 t) + \mathrm{i}\sin(2\pi f_1 t)\right) = h(t)e^{i2\pi f_1}$$

where $h(t)$ is a Gaussian window,
$e^{i2\pi f_1}$ is a complex sisusoid





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Dennis Gabor – Gábor Dénes

- born Budapest, 1900; died London, 1979
- 1918: „főreál” secondary school, Budapest (merged into Dániel Berzsenyi in 1941)
- studied at Technical Universities of Budapest and Berlin
- 1933: flees from Nazi Germany to UK
- 1946: proposes the use of *optimal* (in what sense?) time-frequency atoms to represent signals
- 1947: invents holography at British Thomson-Houston
- 1971: Nobel prize for “his invention and development of the holographic method”

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http://en.wikipedia.org/wiki/Dennis_Gabor

Gabor, Dennis. "Theory of communication. Part 1: The analysis of information." *Electrical Engineers-Part III: Radio and Communication Engineering, Journal of the Institution of* 93.26 (1946): 429-441.

Available to download from bigwww.epfl.ch/chaudhury/gabor.pdf

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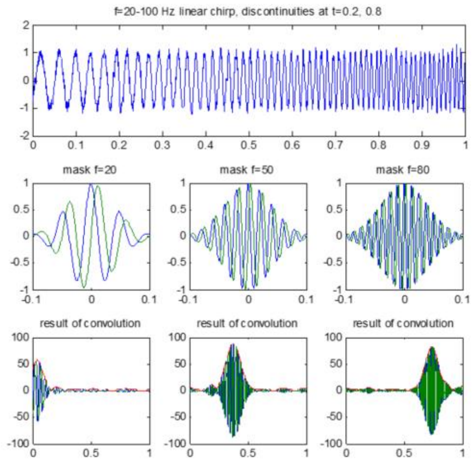
Signal 1: fixed window



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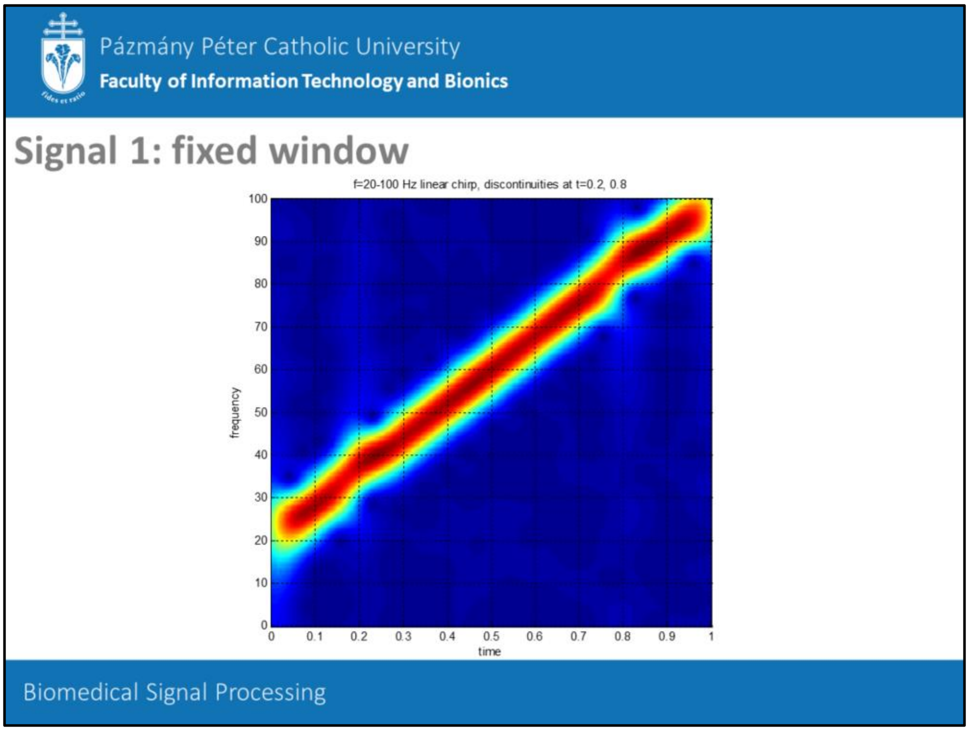
We have a chirp signal (frequency increasing constantly from 20Hz to 100Hz, like a a sound getting higher and higher. In the increase of the frequency there is a discontonuity at 0.2 and 0.8 s.)

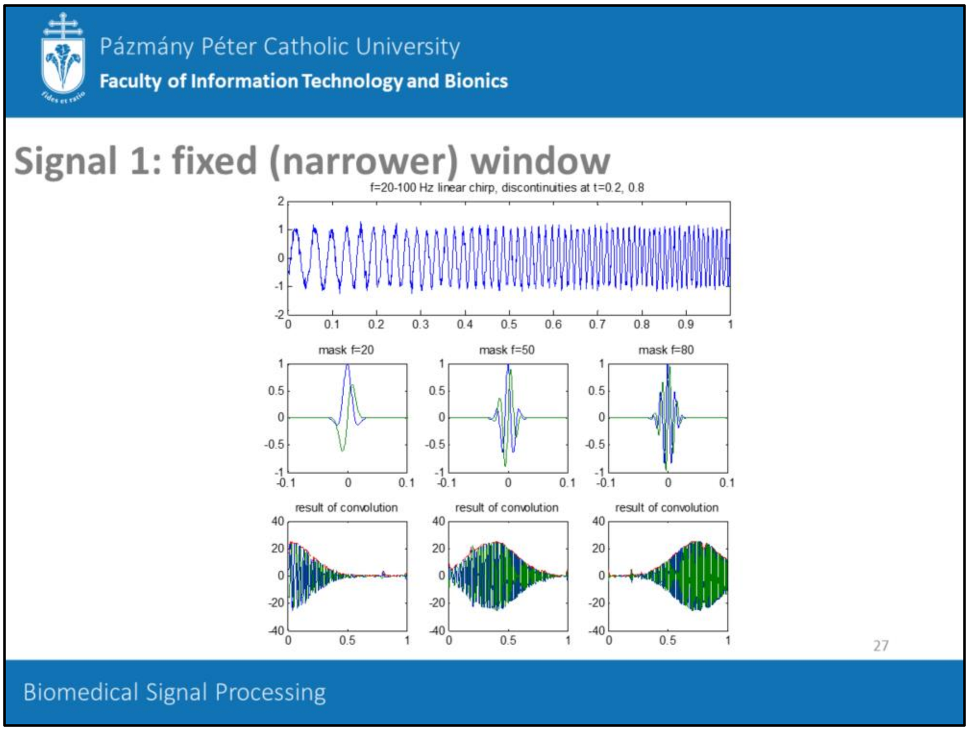
We choose a Gabor-window with different frequencies (20, 50, 80), but same width (0.2)
Green and blue are for the real and imaginary parts of the function

If we convolve the chirp with the different-frequency-Gabors, the resulting amplitude will be the strongest where the frequency of the chirp and Gabor is the most similar.



The previous slide is presented differently.
With the frequency axis the frequency of the Gabor-function is changed. Along the time direction the result of the convolution with the chirp is shown → a spectrogram.

It can be read from the spectrogram that the frequency increases continuously, except for 2 discontinuities at around 0.2 and 0.8 – with this window-size the discontinuity is not sharp.



The chirp signal is the same.
Here the Gabor function has a shorter length (~0.5 s), and the same frequencies (20,50,80 Hz)

Here the convolution shows a broader response – we can not tell precisely, where the frequency of the chirp and Gabor matches.

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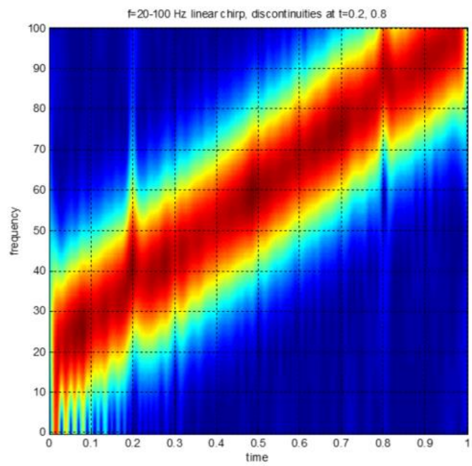
Signal 1: fixed (narrower) window



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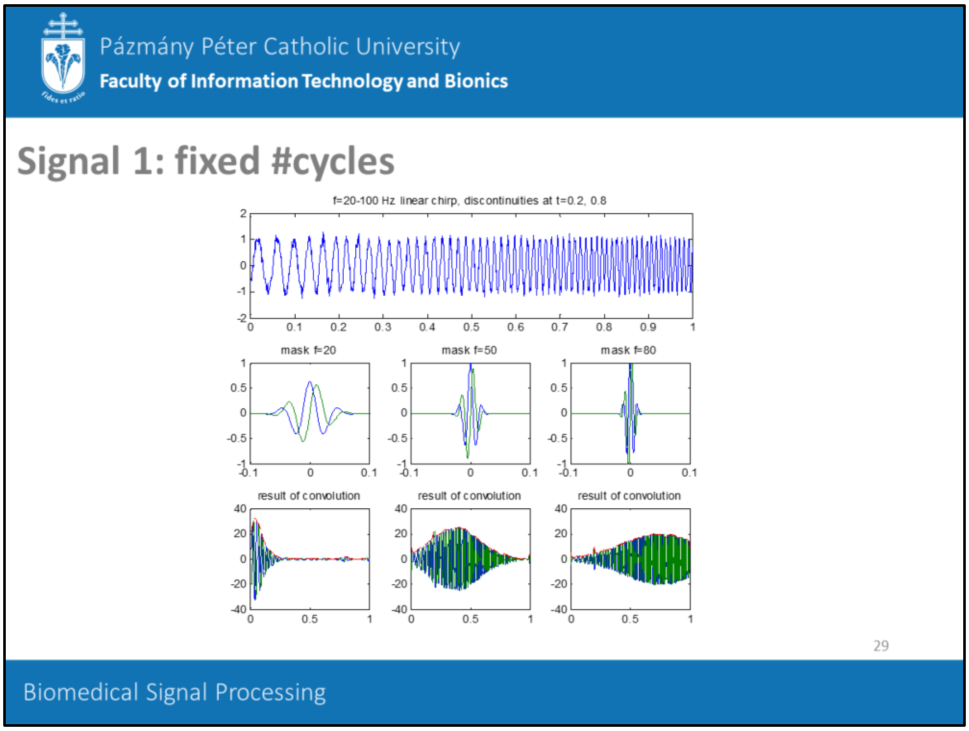
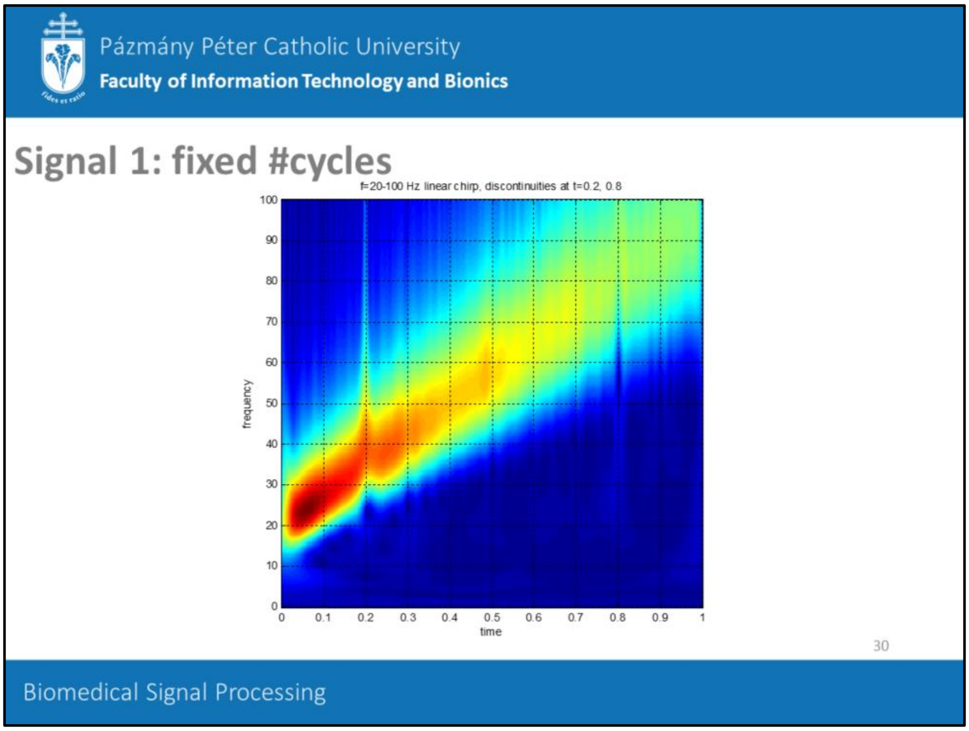
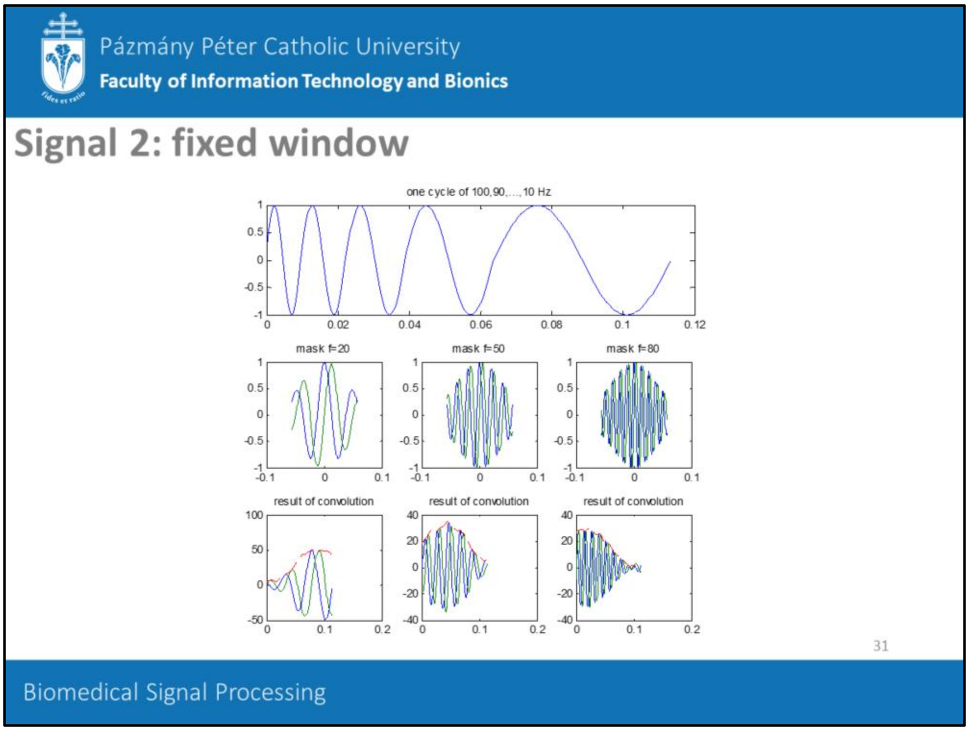
We can see the same here: it is more difficult to tell, what frequency components the signal has at the exact time.
However, the discontinuities are more definite, we can tell exactly, when it occurs.



Now the Gabor functions are changed differently:
The higher frequency is obtained on shorter and shorter windows, fixing the number of cycles.
With the long window the frequency response is sharp, with short window it is smeared.



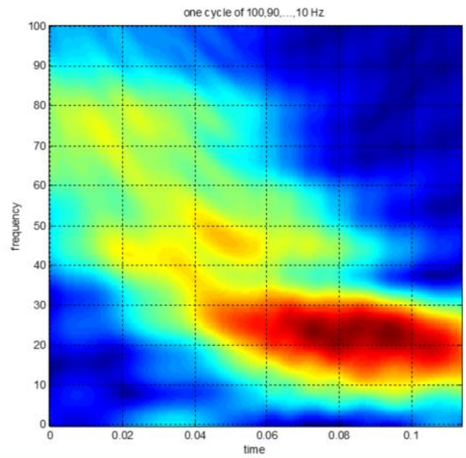
As expected, on low frequencies the time resolution is bad, the frequency resolution is good. On higher frequency it will change: good time and bad frequency resolution.



Instead of continuous change, here the frequency changes from cycle to cyle in discrete steps. Note that now the frequency is decreasing!
It causes that high frequencies have short segments, low frequencies have long segments in the signal.

Now the mask Gabor windows have the same length with the signal, and different frequencies.

Signal 2: fixed window



Remember, in this signal the frequency changes from 100 to 10 in steps of 10, and the corresponting time is longer and longer.
The low frequeny components are easier to localize with the windows, as they are longer in time.

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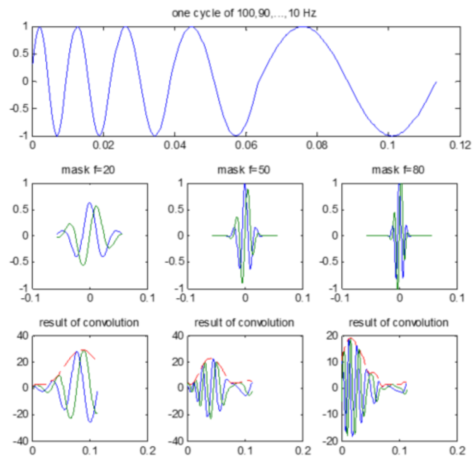
Signal 2: fixed #cycles



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Now again the cyclenumber of the Gabor windows is fixed, having higher frequencies on shorter windows.

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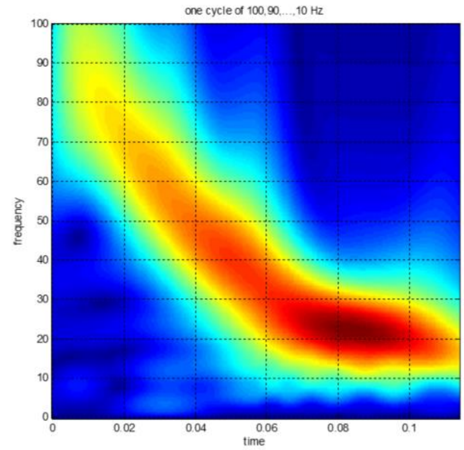
Signal 2: fixed #cycles



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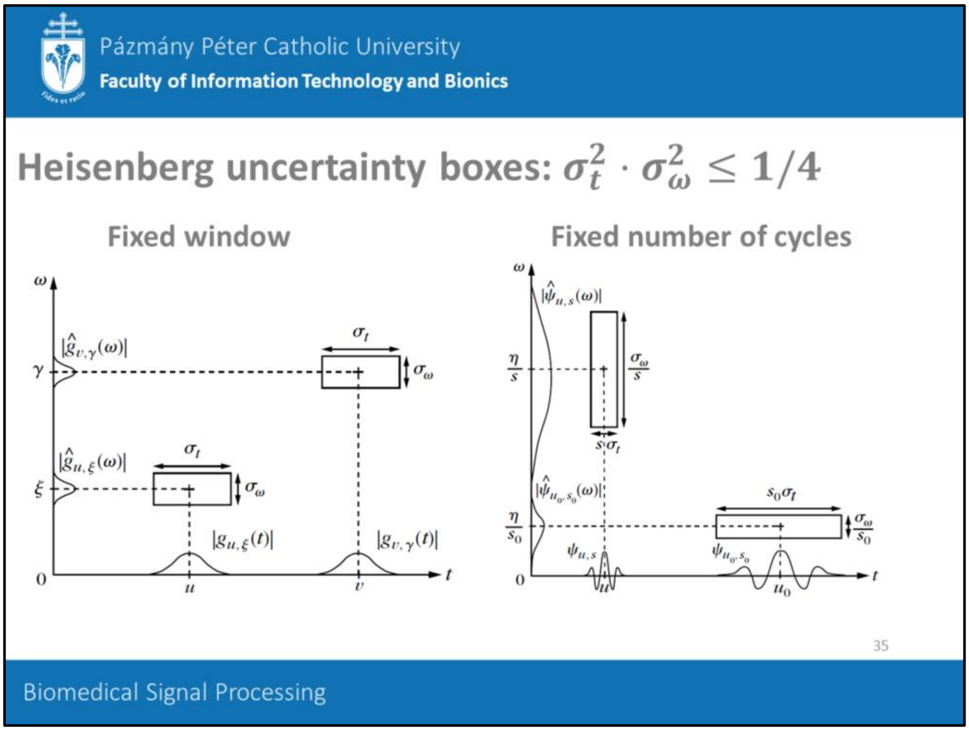
With these windows the frequency change of the signal is well presented. We can see, that the higher frequency components are shorter, the lower ones are longer → crvature of the curve. Furthermore the plot is more defined.



$\boldsymbol{\sigma_t^2 \cdot \sigma_\omega^2 \leq 1/4}$ can be thought of as the area of a rectangle, having sides σ_t and σ_ω can not be larger than a limit.

Fixed window

The standard deviation of the Gaussian function is related to its length, as seen before.

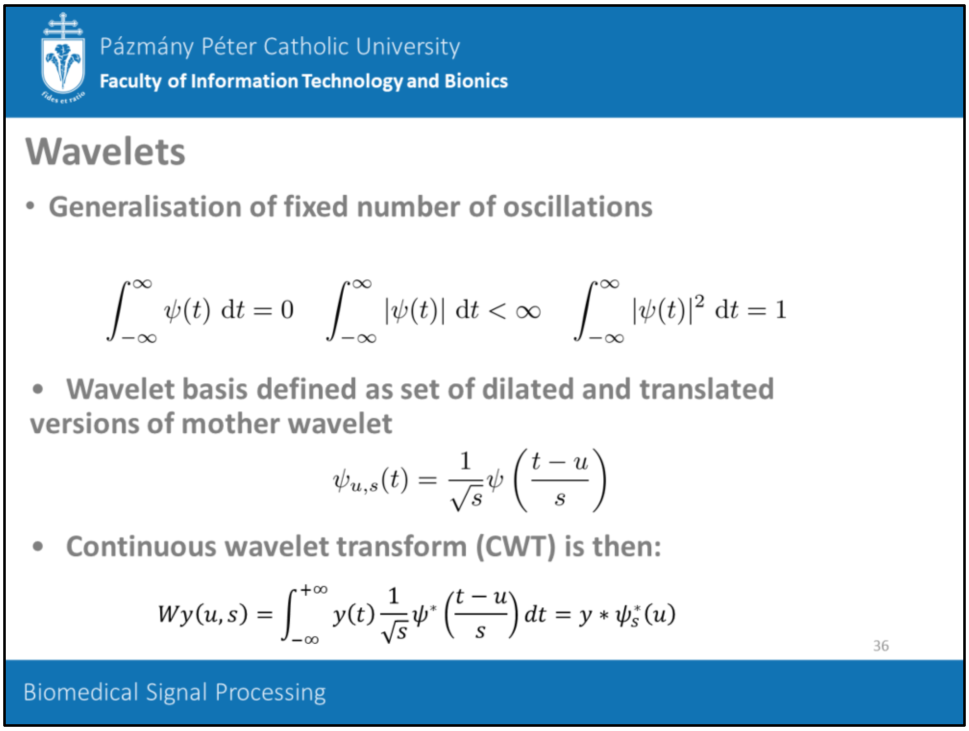
If the time-widow is fixed, the frequency-window (resolution) will also be fixed.

Fixed number of cycles

If the time-window has a fixed number of oscillations, the length of it will change depending on its frequency.

Shorter time-windows will detect fast high-frequency changes with bad f-resolution. Longer windows will not detect fast changes well, but will have nice LF-resolution.

Mallat: A wavelet tour of signal processing pp. 16,18



$\psi(t)$ is a motherwavelet: a function with zero mean, finite ℓ_1 and unity ℓ_2 norms.

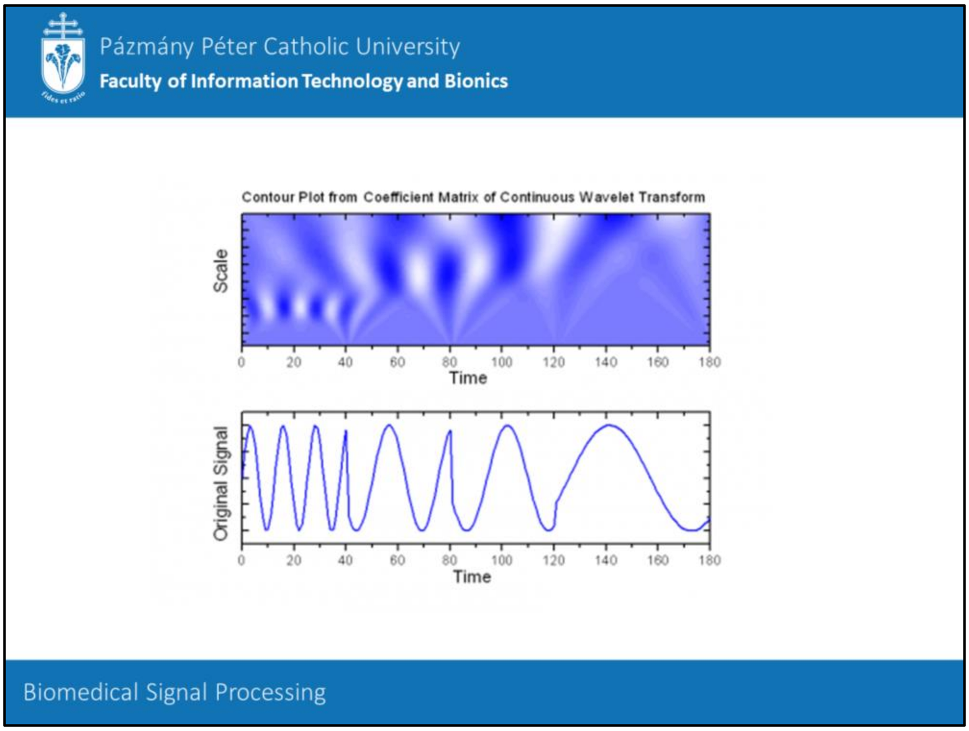
We can define a wavelet-basis from the motherfunction. It will be translated (shifted) with u, and dilated (streched) by 1/s (inverse of central frequency).

The scaling term s ensures that ℓ_2 norm remains 1.

Similar to the Fourier transform, a function y(t) can be expressed as the integral of such wavelets. $\psi^*(t)$ is the complex conjugate. It can be rewritten in the form of a convolution product with

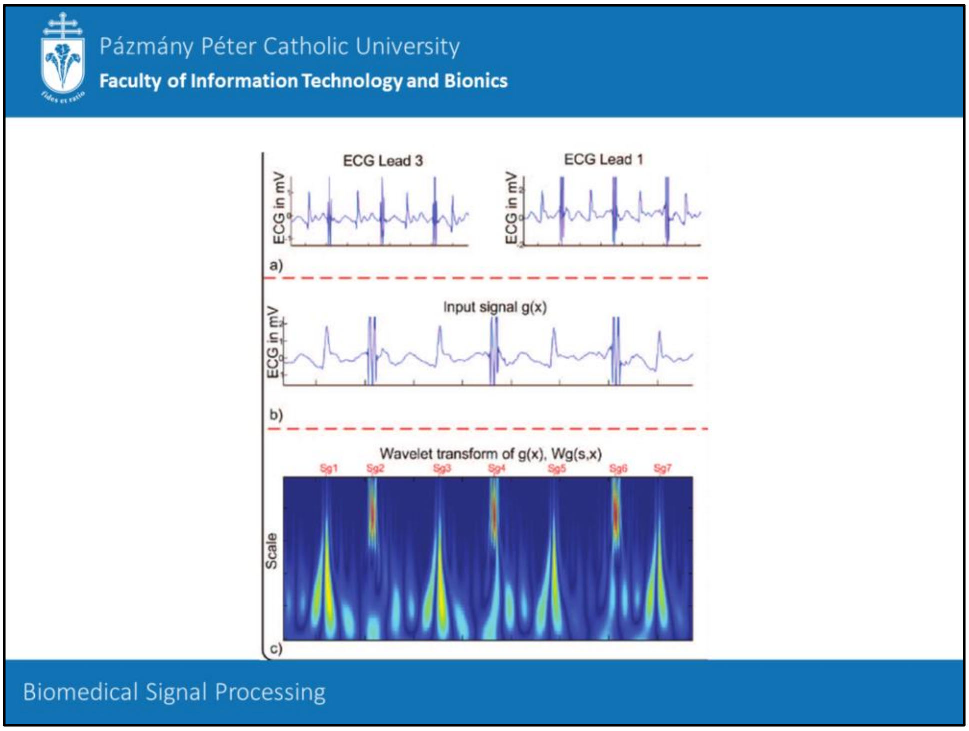
$$\psi_s^*(u) = \frac{1}{\sqrt{s}} \psi^* \left(-\frac{t}{s} \right)$$

Mallat: A wavelet tour of signal processing pp. 3,92,102



With shorter windows (lower rows of the plot) represent the quick changes better.
Long windows on the top have better frequency-resolution.

https://www.originlab.com/doc/Origin-Help/Continuous-WaveTrans



a) Acquisition of two ECG Leads, which correspond to lead 1 and 3 on scheme 1. b) Generation of g (x), which is the projection of the Vectocardiogram on the QRS vector. c) A continuous wavelet transform is applied on g (x). Seven important modulus maximum lines, annotated Sg1 to Sg7 , can be observed on this figure

https://www.researchgate.net/profile/Jacques_Felblinger/publication/224461105/figure/fig2/AS:302732910055433@1449188558784/Description-of-the-presented-method-a-Acquisition-of-two-ECG-Leads-which-correspond.png

Useful references

First look at:

- Matlab Wavelet Toolbox

https://www.mathworks.com/products/wavelet.html

Check out in particular

Remove Time-Localized Frequency Components:
https://www.mathworks.com/help/wavelet/ug/removing-a-time-localized-frequency-component-using-the-inverse-cwt.html

which can be run in MATLAB using the following command (provided you have MATLAB 2017b with Wavelet Toolbox)

openExample('wavelet/ICWTRemovingTimeLocalizedFreqComponentExample')

Student suggestion:

- Wavelet tutorial http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html