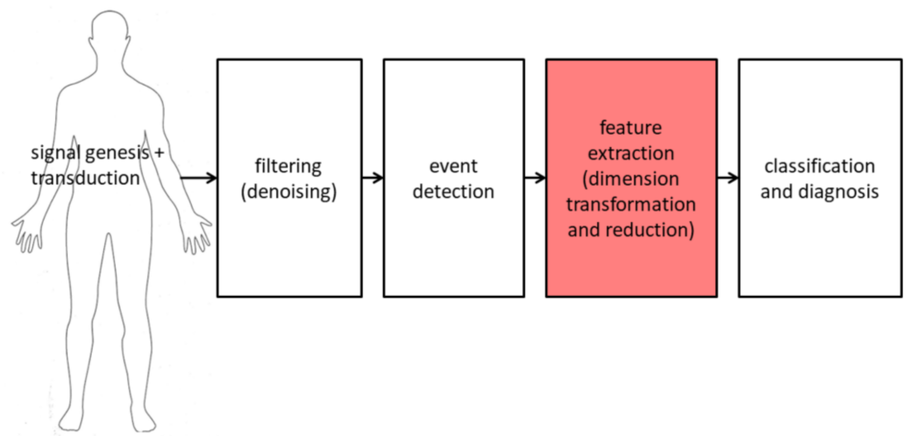
Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

Biomedical Signal Processing
2018-2019 Autumn

Spectral Estimation

Lecturer: Janka Hatvani
Responsible lecturer: dr. Miklós Gyöngy

The BSP Flow Chart



Today's goal

- **What is a spectrum and what can we use it for?**
- **What are the limitations of FFT?**
- **High-resolution spectral estimation methods**
  - Non-parametric (multi-taper, minimum variance)
  - Parametric (AR-model, maximum entropy)
- **Point-processes**

Motivation

- **Estimate event frequency (HR, breathing rate)**
- **Identify frequencies of several components, eg. for filtering**
- **Measure power content in spectral band**
  - HRV: HF/LF ratio
  - EEG: alpha, beta, gamma, etc
- **Estimate pathologies**

What is a spectrum?

„The entire range of wavelengths of electromagnetic radiation”

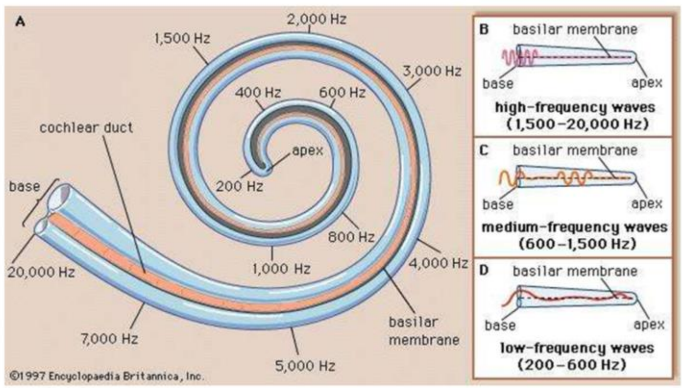
„The components of a sound or other phenomenon arranged according to such characteristics as frequency, charge, and energy”

„A spectrum is a condition that is not limited to a specific set of values but can vary, without steps, across a continuum”

https://en.oxforddictionaries.com/definition/spectrum
https://en.wikipedia.org/wiki/Spectrum

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

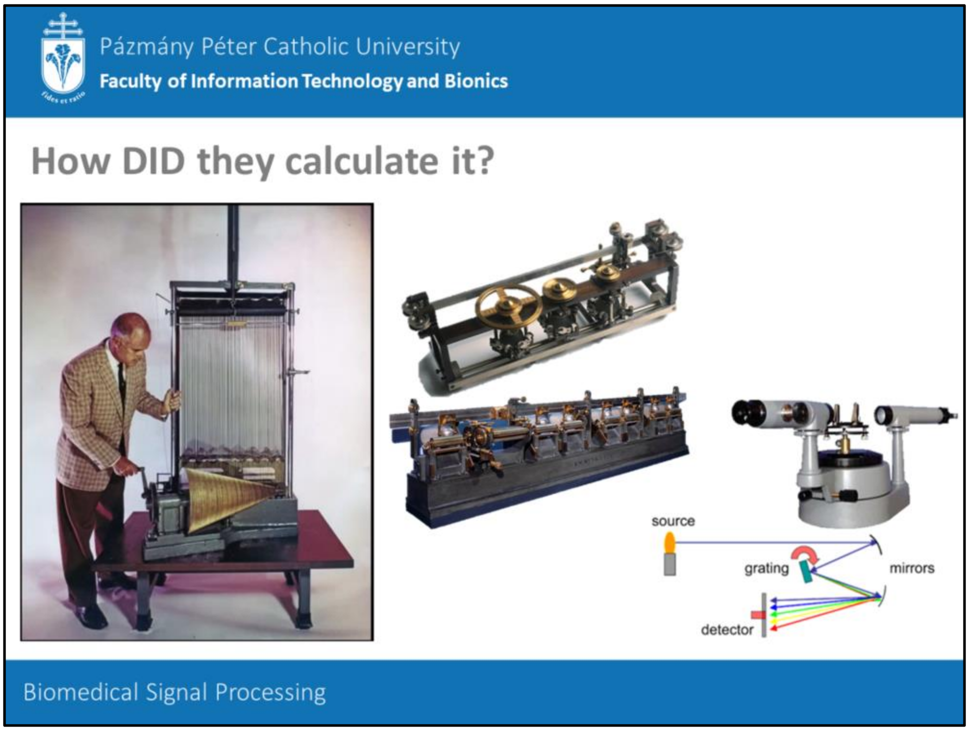
How do WE do it?



Biomedical Signal Processing

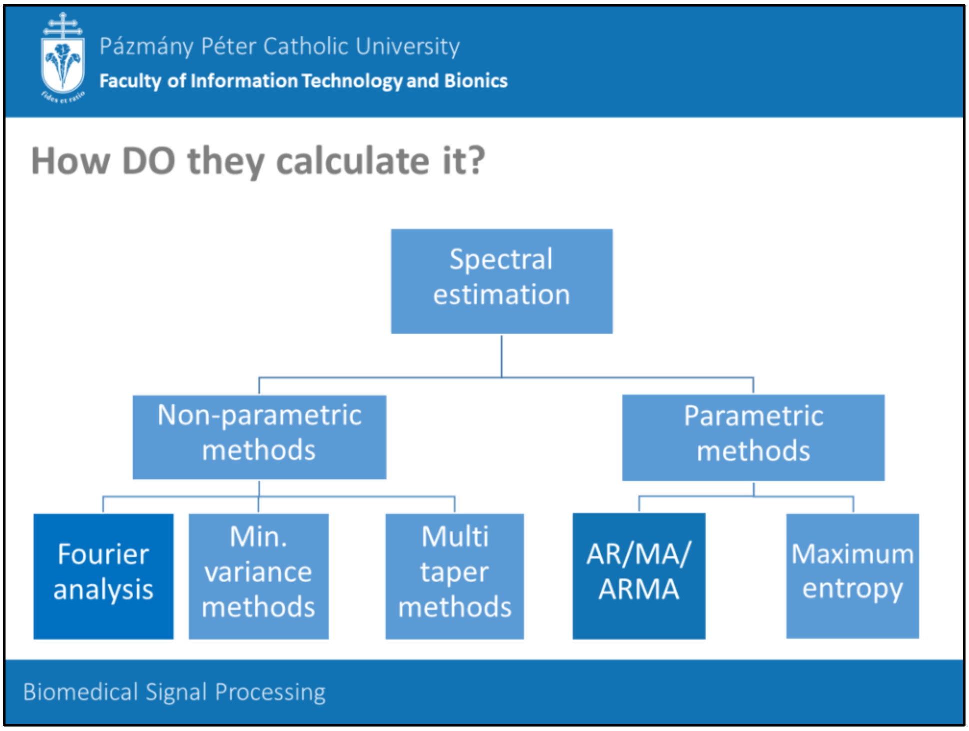
In old times analog computers were built for obtaining the spectral density. These are beautiful machines (if you liked the movie, The Timemachine, you will love it), see the video for an example.
Another example is the optical spectrometer, which analyses the frequency spectrum of the incoming light using mirrors and grating plates.

https://www.youtube.com/watch?v=NAsM30MAHLg
https://en.wikipedia.org/wiki/Optical_spectrometer
http://www.mast.queensu.ca/~driegert/papers/History_of_Spectrum_Estimation.pdf
http://www.ssplprints.com/image/82507/henricis-harmonic-analyser-no-3-1894
https://collection.sciencemuseum.org.uk/objects/co60669/kelvins-harmonic-analyser-harmonic-analyser



In old times analog computers were built for obtaining the spectral density. These are beautiful machines (if you liked the movie, The Timemachine, you will love it), see the video for an example.
Another example is the optical spectrometer, which analyses the frequency spectrum of the incoming light using mirrors and grating plates.

https://www.youtube.com/watch?annotation_id=annotation_2407208685&feature=iv&src_vid=NAsM30MAHLg&v=6dW6VYXp9HM
https://www.youtube.com/watch?v=NAsM30MAHLg
https://en.wikipedia.org/wiki/Optical_spectrometer
http://www.mast.queensu.ca/~driegert/papers/History_of_Spectrum_Estimation.pdf
http://www.ssplprints.com/image/82507/henricis-harmonic-analyser-no-3-1894
https://collection.sciencemuseum.org.uk/objects/co60669/kelvins-harmonic-analyser-harmonic-analyser



- The **non-parametric** approaches explicitly estimate the spectrum of the process without assuming that the process has any particular structure.
    - Fourier – to be discussed later
    - Multi-taper: „each taper (window) is pairwise orthogonal to all other windows. The windowed signals provide statistically independent estimates of the underlying spectrum. The final spectrum is obtained by averaging over all the windowed spectra”
    - Min. variance methods (e.g. Barlett’s method): It provides a way to reduce the variance of the spectrum in exchange for a reduction of resolution. A final estimate of the spectrum at a given frequency is obtained by averaging the estimates from the periodograms (at the same frequency) derived from a non-overlapping portions of the original series
- The **parametric** approaches assume that the underlying stationary stochastic process has a certain structure that can be described using a small number of parameters (for example, using an auto-regressive or moving average model)
    - AR/ MA/ ARMA: to be discussed later
    - Maximum entropy: The method is based on choosing the spectrum which corresponds to the most random or the most unpredictable time series whose autocorrelation function agrees with the known values.

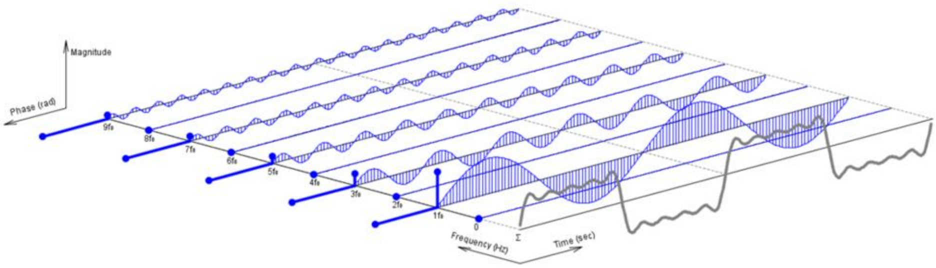
https://en.wikipedia.org/wiki/Spectral_density_estimation

NON-PARAMETRIC

Fourier

About the Fourier *series*

Any periodic signal can be decomposed into a sum of sinusoids at the harmonics of the fundamental period (f_0) → sinusoids of discrete frequencies



http://mathworld.wolfram.com/images/eps-gif/FourierSeriesSquareWave_800.gif

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

Orthogonality of sines

For $m, n \in \mathbb{N}_0$ and fundamental angular frequency $\omega = \frac{2\pi}{T}$:

$$\int_0^T \sin(m\omega t)\sin(n\omega t)\ \mathrm{d}t = \frac{1}{2}\int_0^T \cos((m-n)\omega t) - \cos((m+n)\omega t)$$
$$= \begin{cases} 0.5T & \text{if } m = n \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$\int_0^T \cos(m\omega t)\cos(n\omega t)\ \mathrm{d}t = \frac{1}{2}\int_0^T \cos((m-n)\omega t) + \cos((m+n)\omega t)$$
$$= \begin{cases} 0.5\mathrm{T} & \text{if } m{=}n, \\ \mathrm{T} & \text{if } m = n = 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$\int_0^T \sin(m\omega t)\cos(n\omega t)\ \mathrm{d}t = \frac{1}{2}\int_0^T \sin((m-n)\omega t) + \sin((m+n)\omega t)$$
$$= 0.$$

Biomedical Signal Processing

Orthogonality means that the product of any 2 base functions is zero, except for multiplying with itsel, when it is one.
Why do we integrate from zero to one? → periodic functions, it is enough to integrate over one period

Orthogonality of sines

Using orthogonality property, any function y with periodicity T – or within a range $x \in [0, T]$ – can be expressed as a Fourier series of sinusoids:

$$y = \sum_{n=0}^{\infty} \{a_n \sin(n\omega t) + b_n \cos(n\omega t)\}$$

$$a_n = \frac{2}{T} \int_0^T y \sin(n\omega t)$$

$$b_n = \frac{2}{T} \int_0^T y \cos(n\omega t)$$

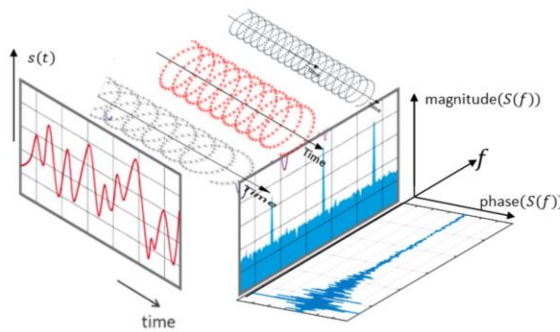
Question: What are a_0, b_0?

$a_0 = 0,$

$b_0 = \frac{2}{T} \int_0^T y(t) cos(0)\, dt = \frac{2}{T} \int_0^T y(t) dt$

About the Fourier *transform*

Any signal can be decomposed into an integral of complex sinusoids at a spectrum of frequencies



https://i.stack.imgur.com/2RX84.png
https://upload.wikimedia.org/wikipedia/commons/6/61/FFT-Time-Frequency-View.png

From Fourier series to the Fourier transform

Taking $T \to \infty$ and using the complex sinusoid to get complex coefficients of each frequency component yields the Fourier transform:

$$y(t) = \int_{-\infty}^{\infty} Y(f) \exp(j2\pi f t) \, \mathrm{d}f$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) \exp(-j2\pi f t) \, \mathrm{d}t$$

In actual practice, the limit of integration will be finite. In the case of periodic functions with a fundamental frequency $1/T$, an integration range of $[0, T]$ can still be used to yield the correct estimates of the coefficients at the frequencies of interest.

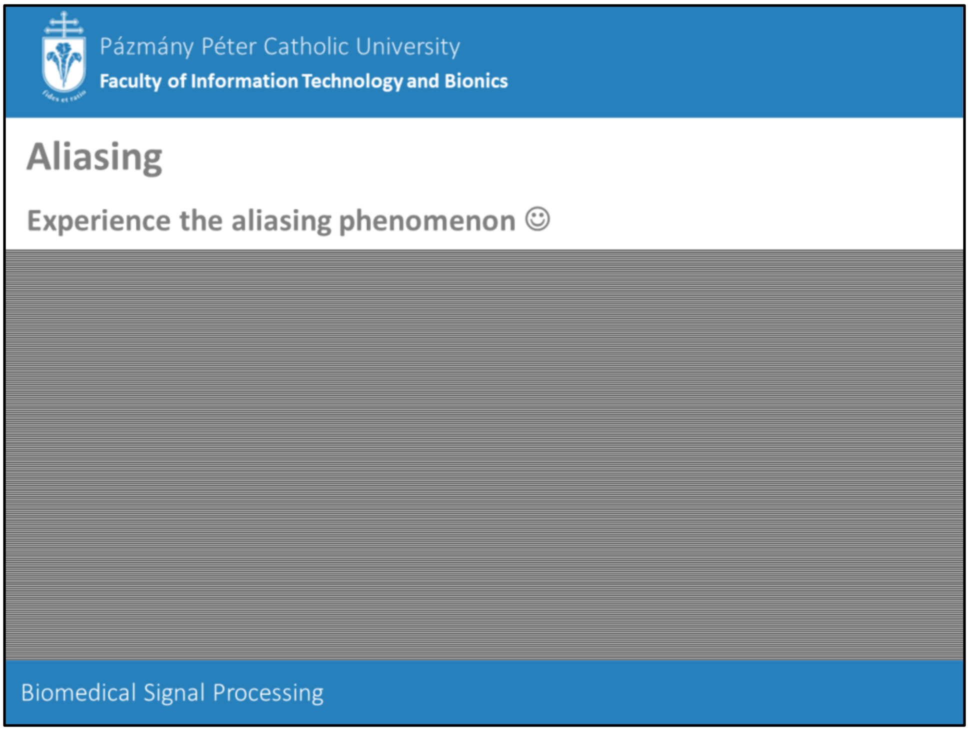
Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

FFT issues

- Aliasing
- Spectral leakage, resolution
- Time-frequency uncertainty
- Complexity (?)

Biomedical Signal Processing

Time-frequency uncertainity will be further discussed following lectures (STFT)
Is it complex? Since FFT became popular, it is rather easy…

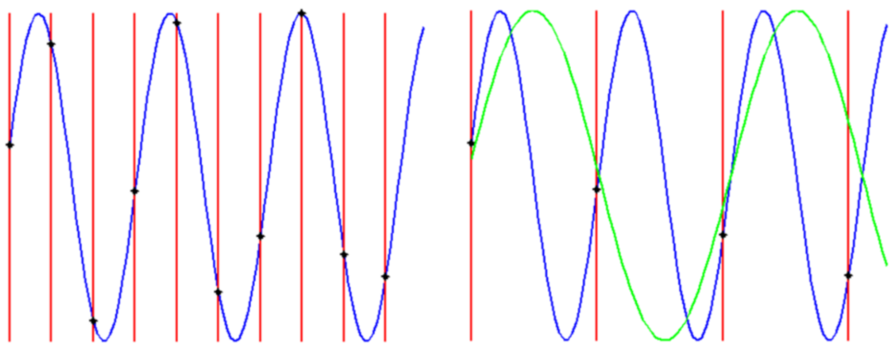


The lines in the image repeat with a frequency. If you come closer/farther from the image with your camera, there will be a distance, from where the spacing of the dector elements in your camera can not cope with this frequency.

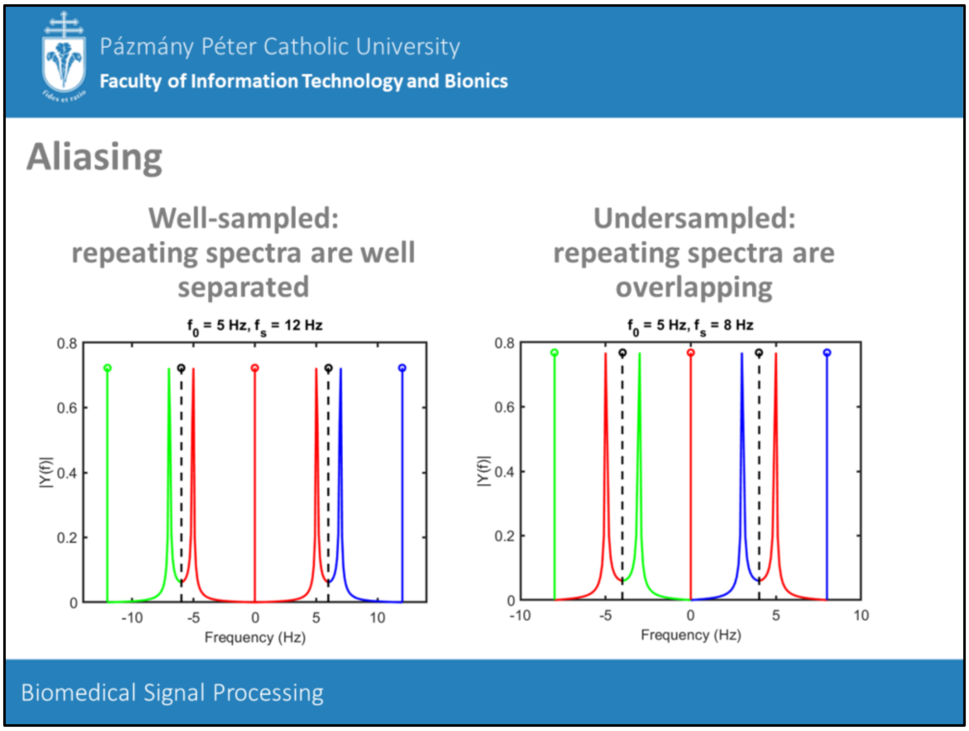
Aliasing

Well-sampled: Reconstruction is OK

Undersampled: Reconstruction is ambiguous

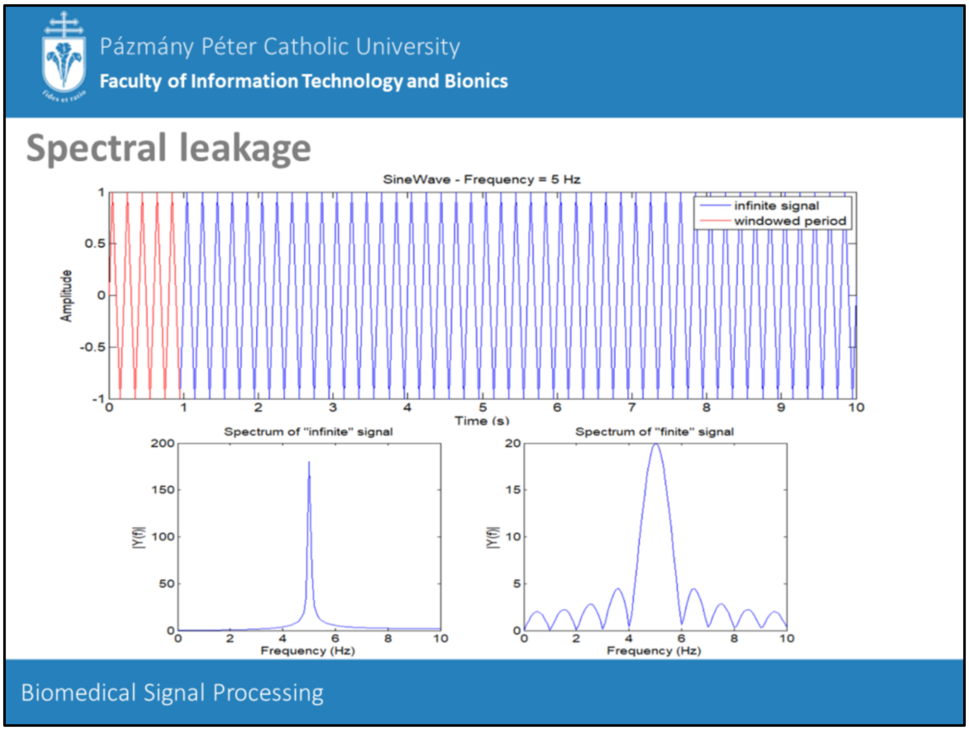


https://www.youtube.com/watch?v=ByTsISFXUoY



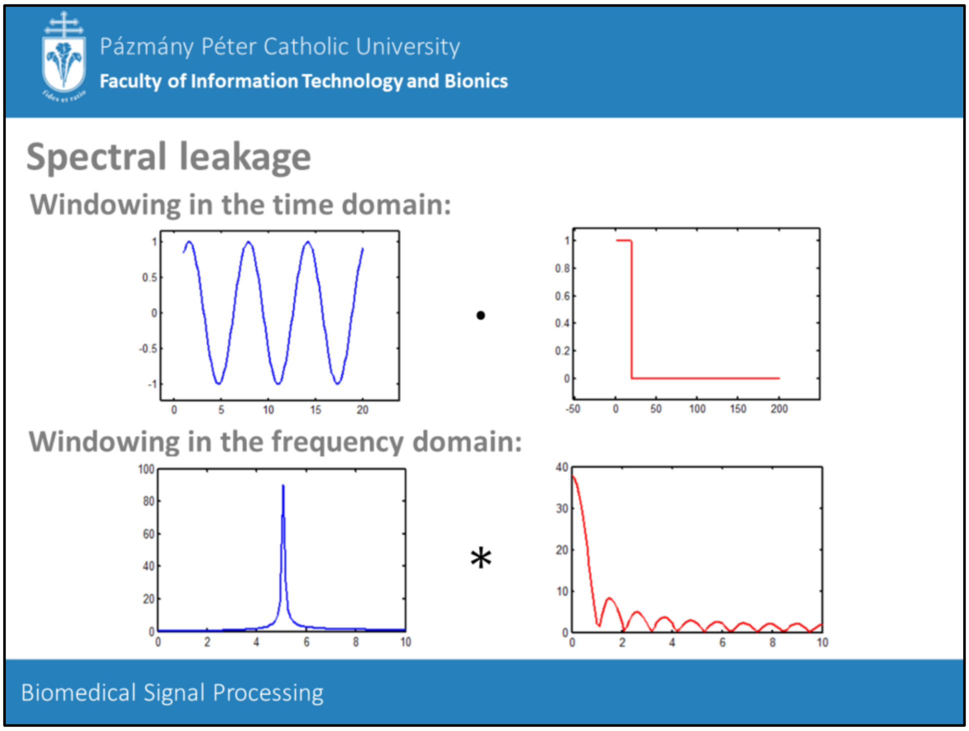
The fft of a discrete signal has a periodically repeating spectrum, where the periodicity is the sampling frequency.
When the nyquist theorem is met ($2f_0 < f_s$), the repeating segments are not overlapping (the colors are well separated). On the other hand, when the sampling frequency is too low, the segments will overlap. When we are applying inverse Fourier, the components of the next segment will corrupt the output signal, new, non-existing frequencies will be considered.

https://www.youtube.com/watch?v=ByTsISFXUoY

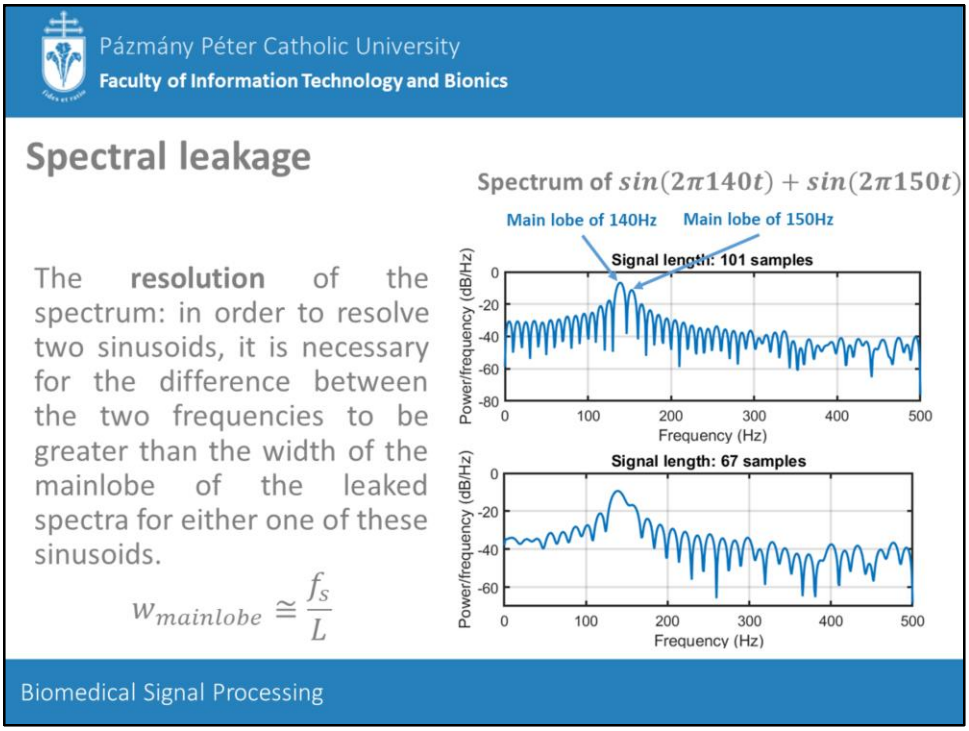


spectral leakage:
The effect of windowing causes leakage – the amount of energy in one bin is distributed across its neighbours.

Ideally even on the left we would see a dirac delta, but even the ‚infinite' signal is not infinite



sleak: both for reduced window, both for zero padded, we have a convolution with sinc function



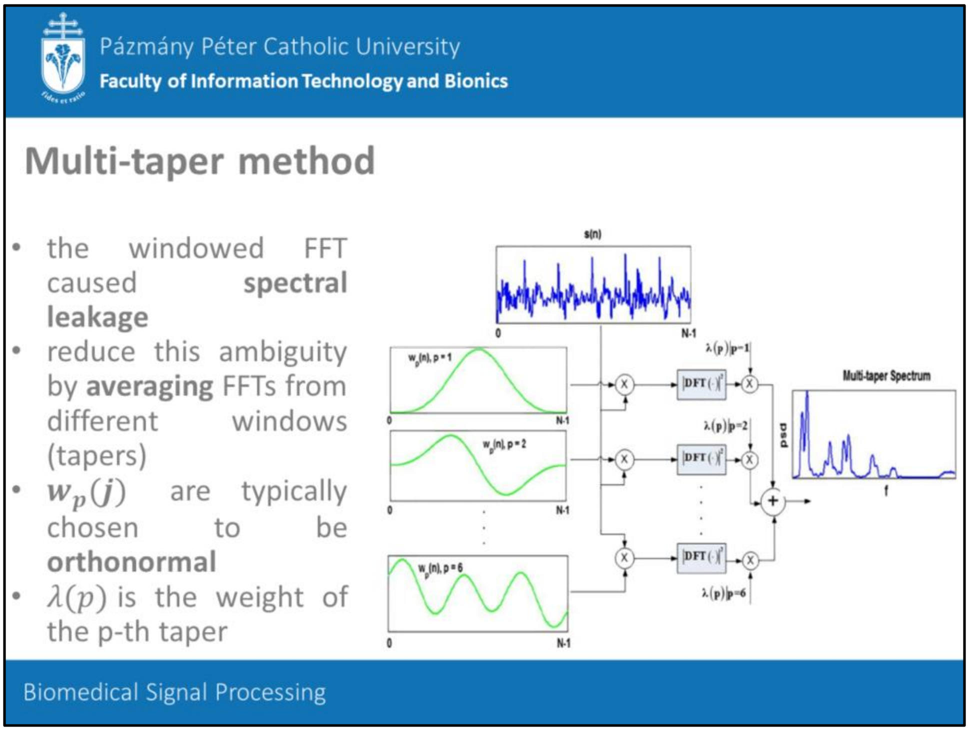
The width of the main lobe is the width where the peak value drops to half of its value. For this kind of functions this roughly equals sampling frequency over sample number (In case of low SNR, it will be even wider)

The signal above is sampled at 1000Hz. The difference between the two frequencies is 10 Hz. In order to resolve them,

http://matlab.izmiran.ru/help/toolbox/signal/spectra7.html

Multi Taper

(improving Fourier)



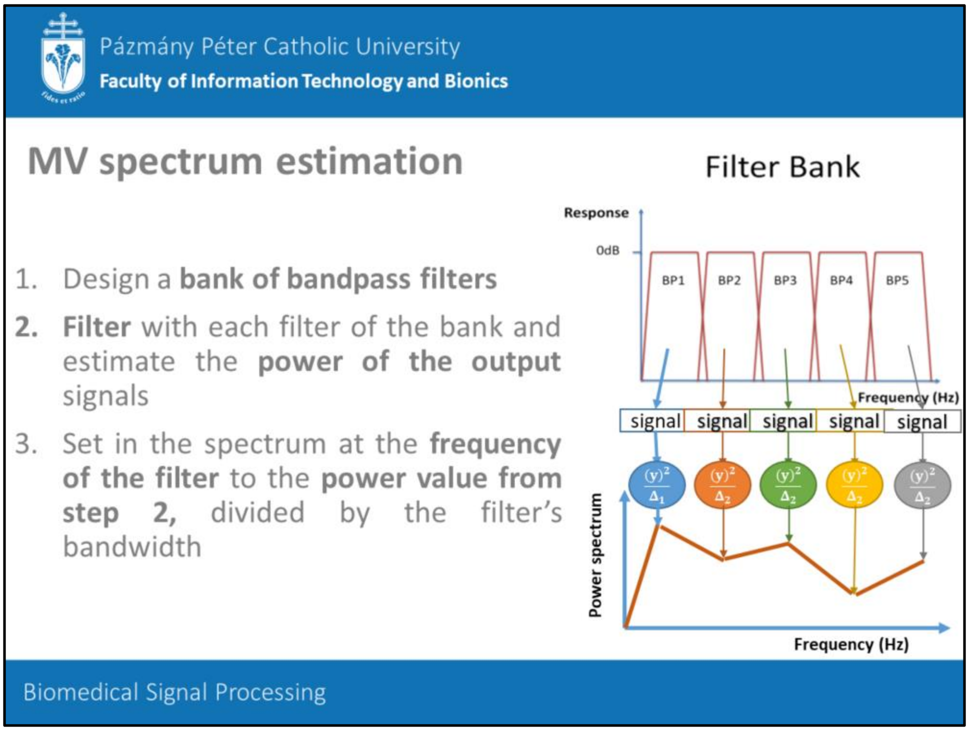
When calculating the FFT, we were windowing, which led to specral leakage (the variance of the estimated spectrum was high compared to the ideal spectrum). This variance should be reduced.

In the Multi-taper method we window (taper) the signal with different windows. Each taper (window) is pairwise orthogonal to all other windows. The windowed signals provide statistically independent estimates of the underlying spectrum. The final spectrum is obtained by averaging over all the windowed spectra

The weighted average of the *M* individual spectral estimates has smaller variance than that of the single-window spectrum estimate by a factor that ap-proaches 1/M (remember, how synchronized averaging improved the SNR by a factor of M)

(PDF) Speech recognition in reverberant and noisy environments employing multiple feature extractors and i-vector speaker adaptation. Available from: https://www.researchgate.net/publication/282951992_Speech_recognition_in_reverberant_and_noisy_environments_employing_multiple_feature_extractors_and_i-vector_speaker_adaptation [accessed Oct 12 2018].

Minimum Variance
spectrum estimation



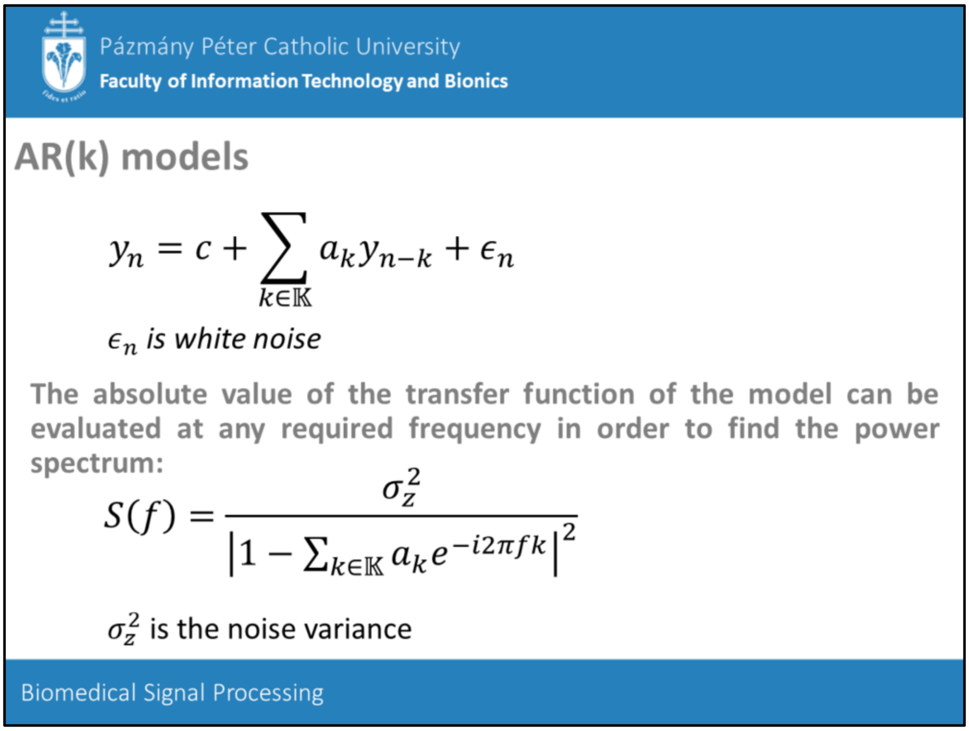
- The bandpass should be as narrow as possible, with 0dB in the pass and high attenuation in the stop band
- The power of the output signal is usually calculated from its autocorrelation

The larger the order of the filters, the better the resolution of the spectrum will be (upper bound on order is the samplenumber!)

https://www.slideserve.com/leroy-lawrence/8-3-minimum-variance-spectrum-estimation

PARAMETRIC

AR models

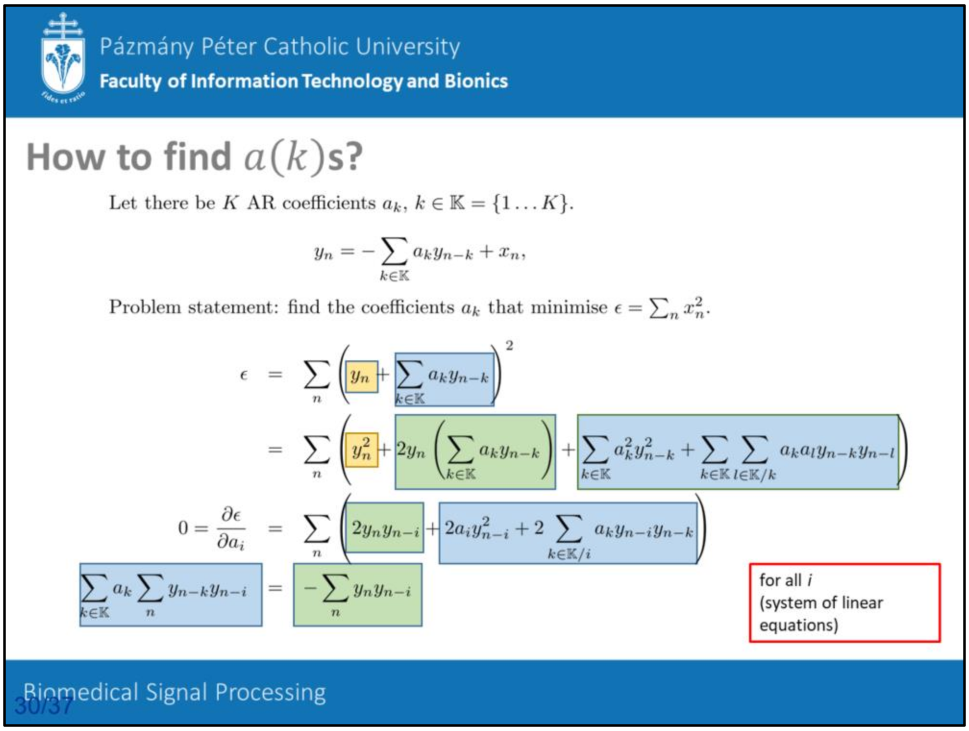


AR model means, that the output of a system can be written as the linear combination k previous outputs (the signal depends only on its own output) and some white noise. It means that the system will have only poles, no zeros. (Note, that the above equation is a simple difference equation, a thing that is very familia to us by now)

- a_k-s are the coefficients of the linear combination, y_{n-k} is the output from k steps before.
- In this case the power spectral density can be calculated using the above equation.

Further models:

- *the MA model is used when spectra are well characterised by their notches*
  MA means moving average, the output can be described as the linear conbination of n previous inputs and some white noise
- *The AR model is effective for spectra that are well characterised by their peaks (resonances)*
- *ARMA model is effective when spectra to be estimated contain both sharply defined peaks and notches at particular frequencies*
  ARMA processes combine the above 2.



We would like to describe the output (y_n) in such a way that it depends only on K previous outputs ($y_{n\_k}, k \in K$). The effect of the current input (x_n) is only like a small error (ϵ)

- We sum the square of the error function over all n
- $(a + \sum_{k\in\mathbb{K}} b_k)^2 = a^2 + 2a\sum_{k\in\mathbb{K}} b_k + \sum_{k\in\mathbb{K}}\sum_{l\in\mathbb{K}} a_k a_l = a^2 + 2a\sum_{k\in\mathbb{K}} b_k + \sum_{k\in\mathbb{K}}\sum_{l\in\mathbb{K}/k} a_k a_l + \sum_{k\in\mathbb{K}} a_k$
  from the square of the sum the a_k^2 s are collected separatedly, $\mathbb{K}/k$ means that from the set $\mathbb{K}$ k is taken out
- In the next step the equation is derivated by each a_i sequentially, as we are looking for a minimal value of the error function.
  1. y_n^2 does not depend on a_i, its derivate is 0.
  2. From the derivate of the second term sum only the component containing a_i remains
  3. From the third term again only a_i remains
  4. From the last term all components containg a_i remain
- Rearranging the equation. The first term is on the right hand side. The second and the third terms are combined (the second term is merged back into the sum, like in the second line of this explanation. Then $\sum_n$ and $\sum_k$ can be changed, and

a_k can be placed befor the $\sum_n$, as it does not depend on n.

Remember, that it is a system of equations, for all different a_is, as we want to otain all K coefficients.

R p. 334, SL p. 442

R: Rangayyan (2002): Biomedical Signal Analysis: A Case Study Approach
SL: Sörnmo and Laguna (2005): Bioelectric Signal Processing

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

How to find $a(k)$s - autocorrelation method

Problem statement: solve the equation below for unknown a_k

$$\sum_{k\in\mathbb{K}} a_k \sum_n y_{n-k} y_{n-i} = -\sum_n y_n y_{n-i}$$

$$\text{convolution: } (u*v)_k = \sum_n u_n v_{k-n} = \sum_n u_{k-n} v_n$$

$$\text{correlation: } \operatorname{corr}(u,v)_k = \sum_n u_n v_{n-k} = \sum_n u_{n+k} v_n$$

Letting r_k be the autocorrelation of y_n, the topmost equation yields:

$$\sum_{k\in\mathbb{K}} a_k r_{i-k} = -r_i$$

Biomedical Signal Processing

Now we have to solve the preiously obtained system of equations.

The convolution is commutative in the sense, that either functions can be shifted along the other

The correlation function (similarly to the above) can be shifted in either way.

In the **autocorrelation** function the function is shifted along itself, and compared to its original version. In $\mathrm{r_k}$ k means the size of the shift. $\sum_n y_n y_{n-i}$ calculates, that if we shift the signal by k samples compared to itself, how similar they are. If the shift is zero, this is the square of the signal, the maximal value of the correlation. If we shift both the signal to compare, and the signal to be compared to ($\sum_n y_{n-k} y_{n-i}$), it is like shifting only one of them by i-k samples ($\sum_n y_n y_{n-(i-k)}$).

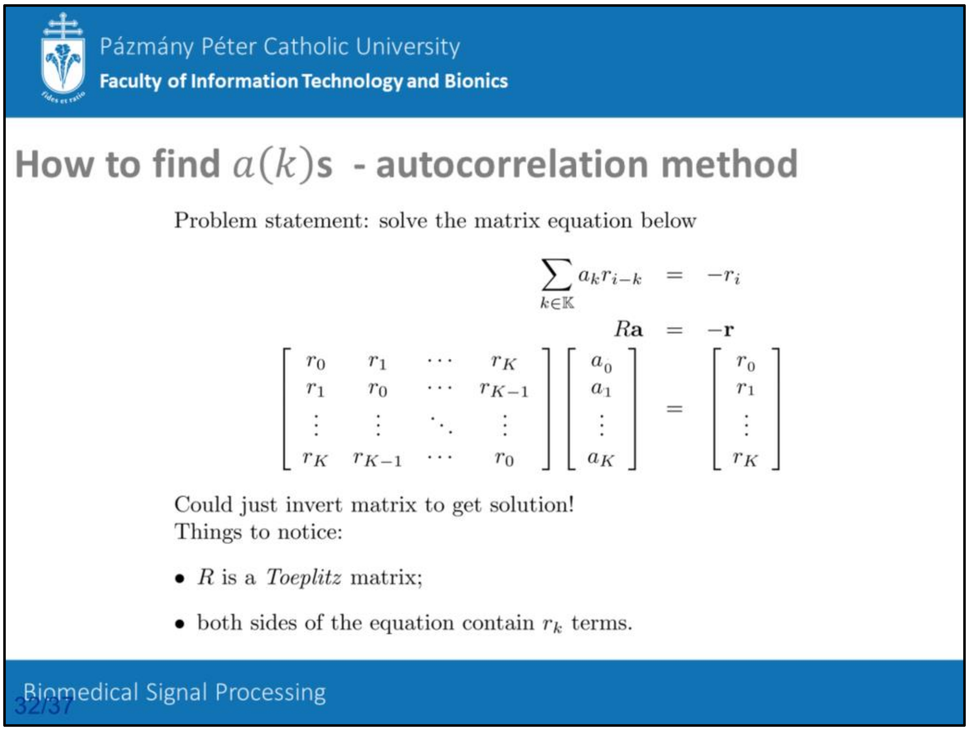
The signal above is extended by **padding with zeros** whever the argument demands more samples for calculating the autororrelation.

The system of equations can be rewritten in the below form, using the autocorrelation functions.

R p. 334, SL p. 442

R: Rangayyan (2002): Biomedical Signal Analysis: A Case Study Approach

SL: Sörnmo and Laguna (2005): Bioelectric Signal Processing

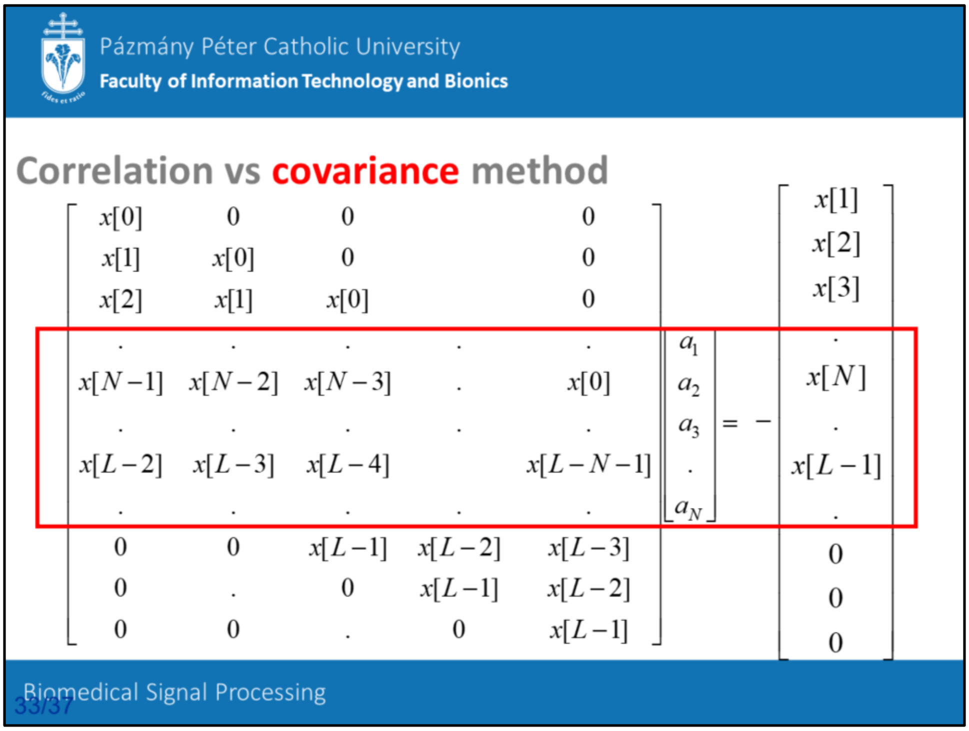


The **autocorrelation matrix** is computed directly from the given signal (easy to obtain).
Afterwards obtaining a(k)s is a simple inversion: a = $-R^{-1}r$
Inversion is usually highly costly an operation, we have to find simplifications.

Toeplitz matrix (same values along all diagonals): use so-called Levinson-Durbin equation to find solution → computationally efficient

R p. 334, SL p. 442

R: Rangayyan (2002): Biomedical Signal Analysis: A Case Study Approach
SL: Sörnmo and Laguna (2005): Bioelectric Signal Processing



If we take only the **entral part of the autocorrelation matrix** containing no zero padding then we have the Covariance Method.

In the covariance method, only a subset of the total possible rows used in the autocorrelation method is taken

(correlation approach implicitly assumes that outside a window $0 < i < L, \quad y[i] = 0$*, as we used zero padding)*

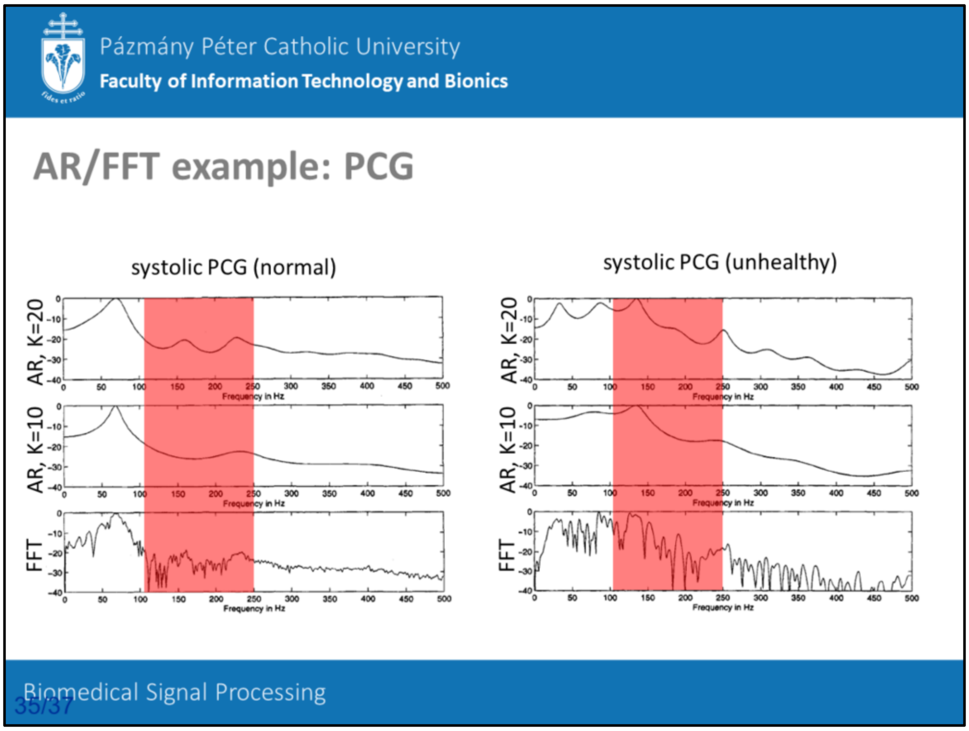
Correlation vs covariance method:
http://www.commsp.ee.ic.ac.uk/~agc/course4_files/Lecture_Slides_0304/6-Spectrum Estimation/3-Modern Spectral Estimation.ppt

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

Comparison of different methods

	Yule-Walker Autocorrelation	Covariance	Modified Covariance	Burg
LMS estimation (forward – F, bacward – B)	F	F	F&B	F&B
Handles short segments well		+	+	+
Stability	+			+
Avoids line splitting (one frequency component portrayed as several at high model orders)			+	-
Can extract more frequency components than order of model		+	+	

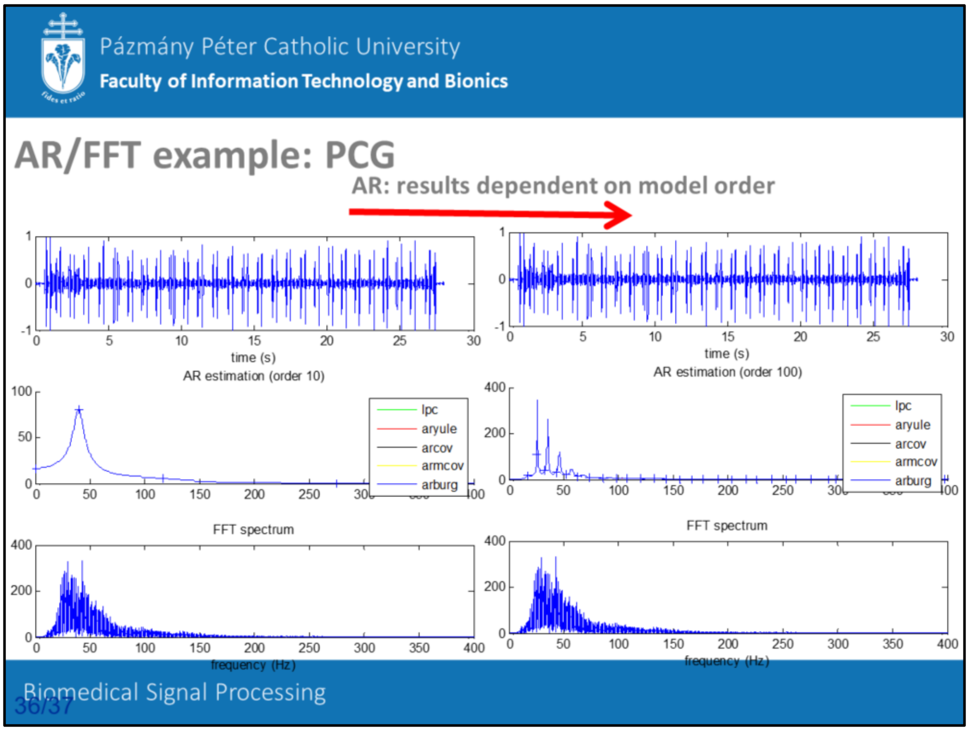
Comparison of different methods:
http://www.mathworks.com/help/signal/ug/spectral-analysis.html#f12-24897



AR spectrum is less effected by noise, has a more definite, smoothed spectrum.

systolic murmur, split S2, opening snap of mitral valve
systolic murmur has given rise to more medium frequency components (highlighted in red)

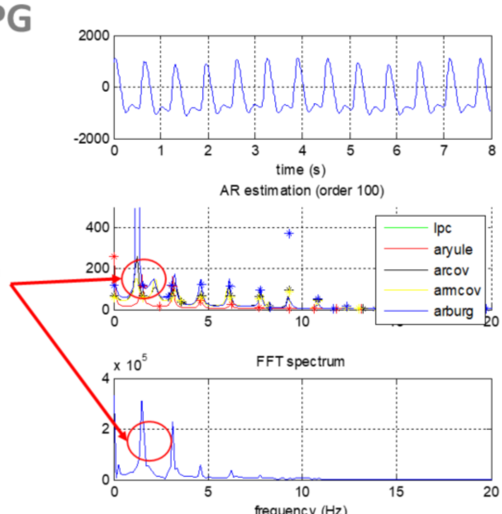
AR/FFT example: PCG (30): R p. 334; *R: Rangayyan (2002): Biomedical Signal Analysis: A Case Study Approach*

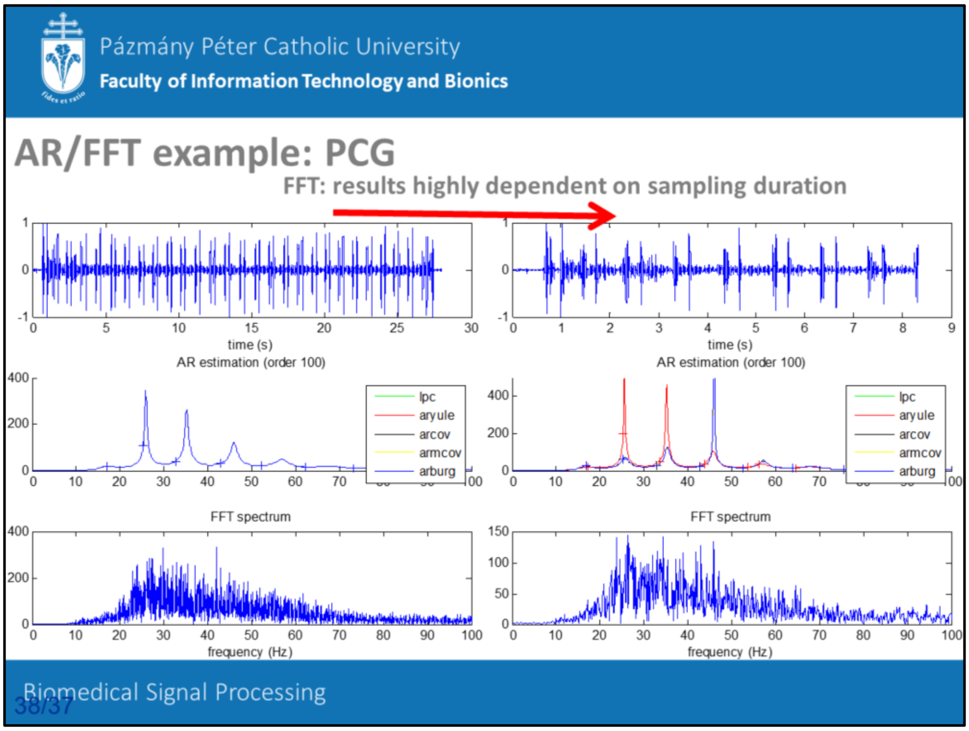


Here you can see the negative side of a parametric method: the parameter has to be estimated

ART/FFT example: PPG

Note the spurious peak with high model order





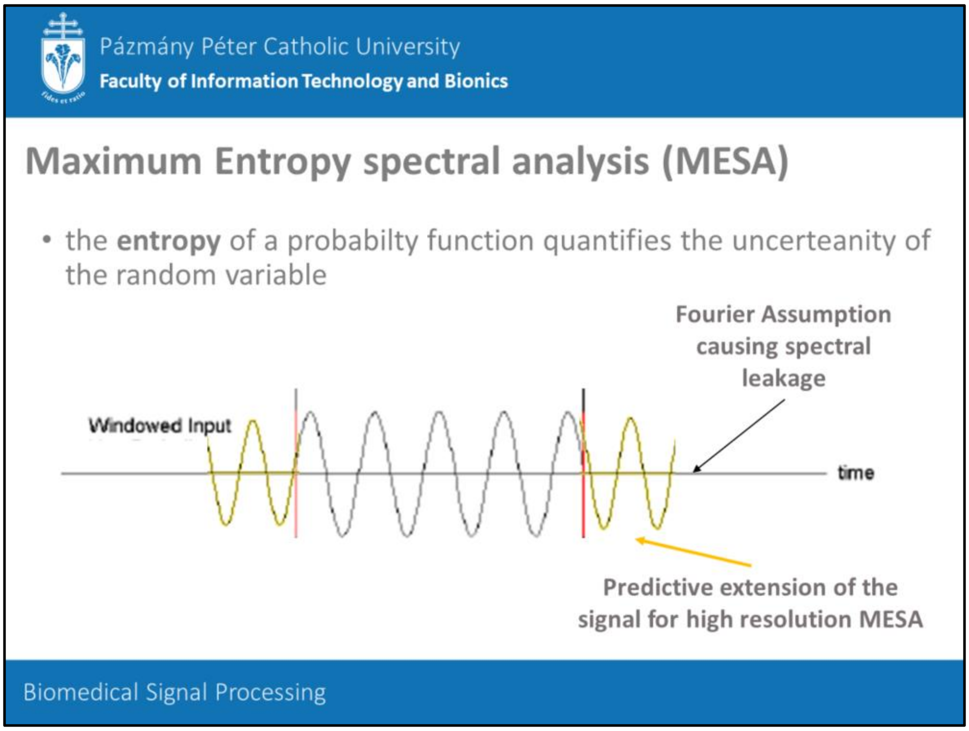
On the other hand, the FFT spectrum of a shorter sample is more ambiguous, however, it does not effect the AR method gravely.

Comparison with FFT

FFT	AR models
+ Non-parametric	- parametric, *k* has to be chosen
+ computationally very efficient	- For high order systems introduces spurious spectral peaks
- Spectral leakage for windowed signals (time-frequency uncertainty)	+ gives good, sharp estimates for short signals too
- Difficult if it is to be evaluated at non-uniform distribution of frequencies	+ Can be evaluated for any frequencies, no spectral leakage
- Buried in noise spectrum with low SNR	- Spectral Line Splitting at high SNR (not all methods)

Maximum Entropy Method

What does spectrum of point process look like?
What if frequency content can be described by simpler model?

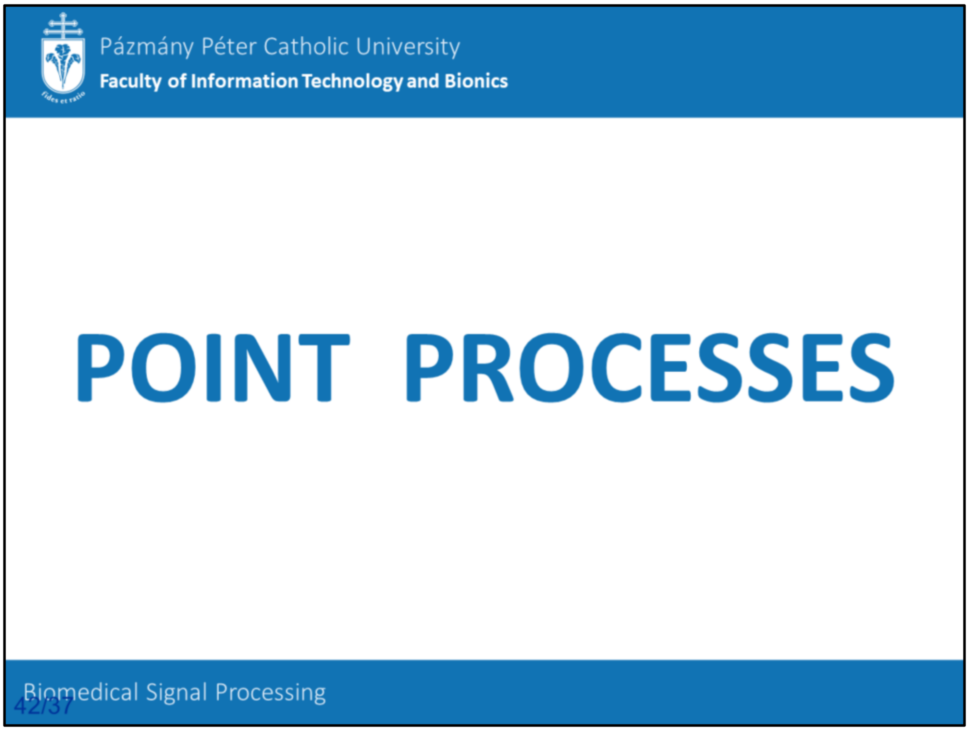


The method is equivalent to:

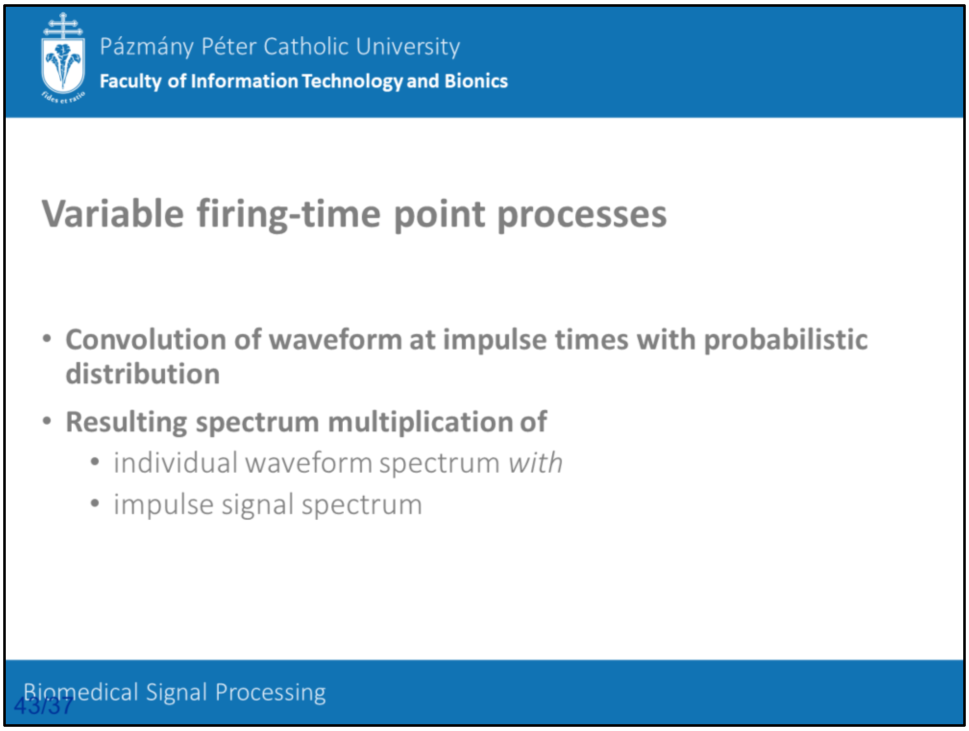
1. Fittin an AR model to the data and finding its spectrum
2. applying a prediction filter to the time series. It will extend your signal. The spectrum of this signal will be the high-resolution spectrum (on the slide)
3. Predict missing parts of your AR sectrum, eg. by extending your autocorrelation function

http://hosting.astro.cornell.edu/~cordes/A6523/A6523_2015_Lecture17.pdf

MESA has one principal advantage over the standard Fourier transform method of spectral analysis: resolution of peaks in the power spectrum is enhanced for short data sequences. MESA has two principal disadvantages: (a) computation time is increased (substantially for long data sequences) and (b) the best choice of for the operator length is not known (poor choices can give misleading results for short data samples).



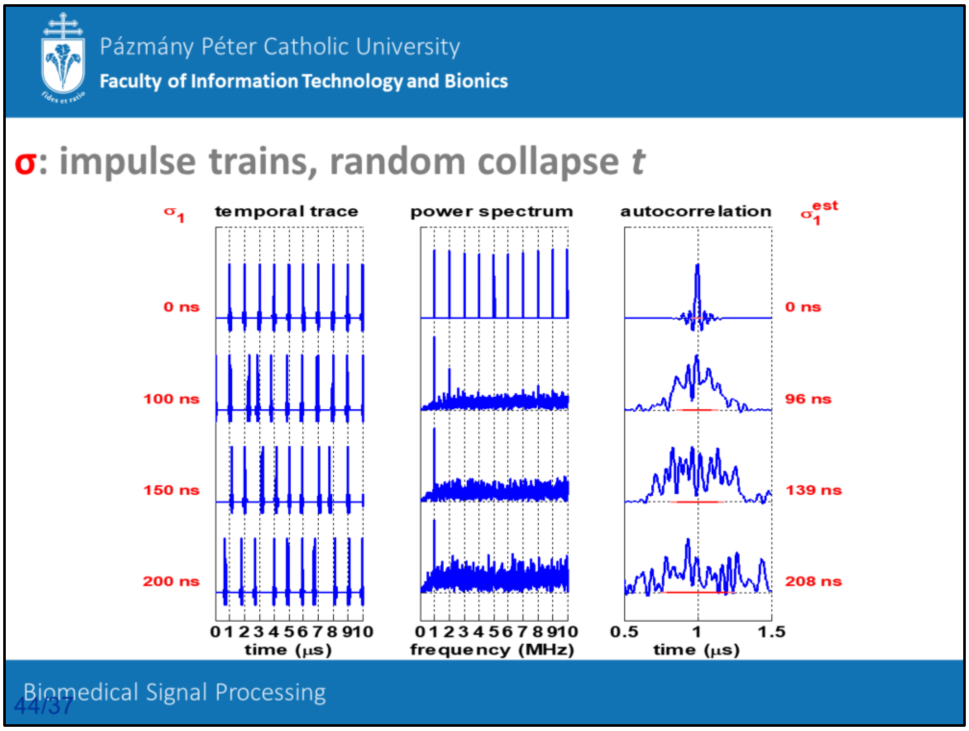
What does spectrum of point process look like?
What if frequency content can be described by simpler model?



So far we have seen the FFT, the AR method, when we assume that the process is autoregressive. Now we will assume that the signal is self-similar, it is composed of the same kind of short events. Such a model is a point-process model.

Point processes can be used as mathematical models of phenomena or objects representable as points in some type of space. At given points something happens – but always the ‚same’ thing with some noise. Like in case of a forest, the location of the trees are the ‚oints’ and the trees are the objects in the space of trees. Or with S1, S2 waves we can talk about the ‚space of sound waves’. (This will be seen again with spare representations)

Biomedical examples: neural spike trains, muscle contractions, heart sounds, ECG TQRSP waves etc.

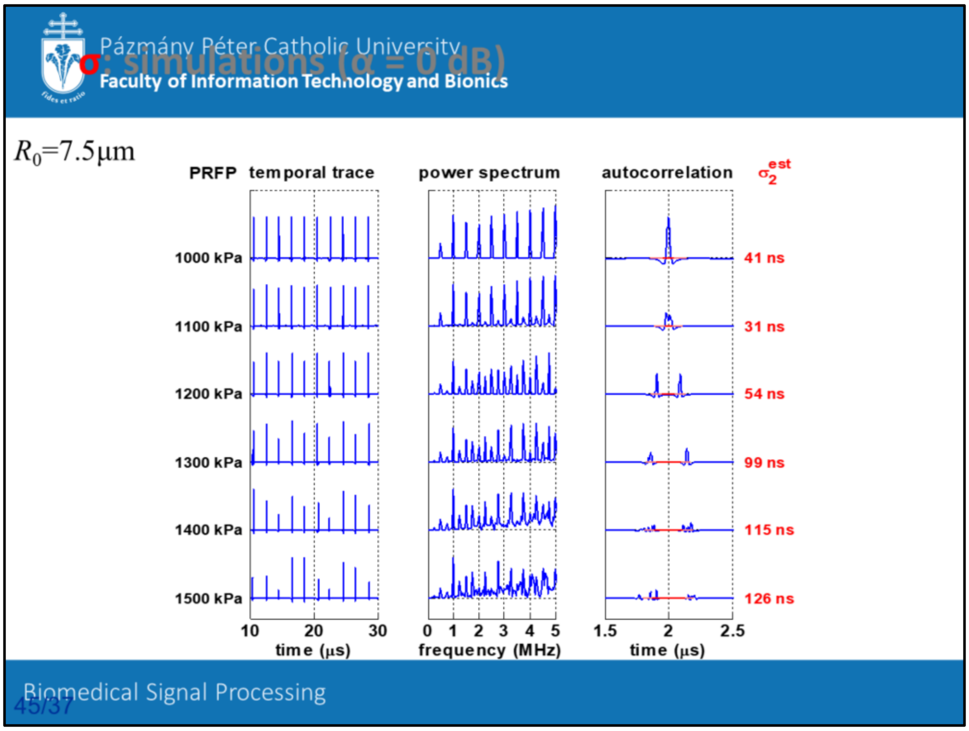


In this plot you can see **cavitation**-induced sound waves. These are very sharp sound waves caused by the resonance-induced collapse of microbubbles. (an external soundwave starts to resonate the bubble on its resonance frequency, leading to its collapse) During Ultrasound imaging you have to consider them for safety issues. https://www.youtube.com/watch?v=QXK2G2AzMTU

The collapse time of the bubble deviates from a regular event every 1μs (the frequency of the external soundwave) with a standard deviation shown on the far left (meaning that bubbles will collapse less and less regularly). So with 0 deviation the collapses happen at exactly 1-1 μs , and with higher deviations this collapse happens within a short time window.
The estimates of this standard deviation, termed “phase instability”, are shown on the right and correspond closely to the true phase instability. It is calculated from the zero-crossing of the signal’s autocorrelation.

With zero deviation the power spectrum shows all harmonics, as the signal is perfectly periodic. The autocorrelation is also very regular.
If the firing times more and more irregular: power spectrum will be more and more broadband, just like the autocorrelation function.
Since there is still an average period between spikes, the fundamental component still appears on the FFT plot (first high peak).

Gyöngy and Jensen (2013): Characterization of cavitation based on autocorrelation of acoustic emissions

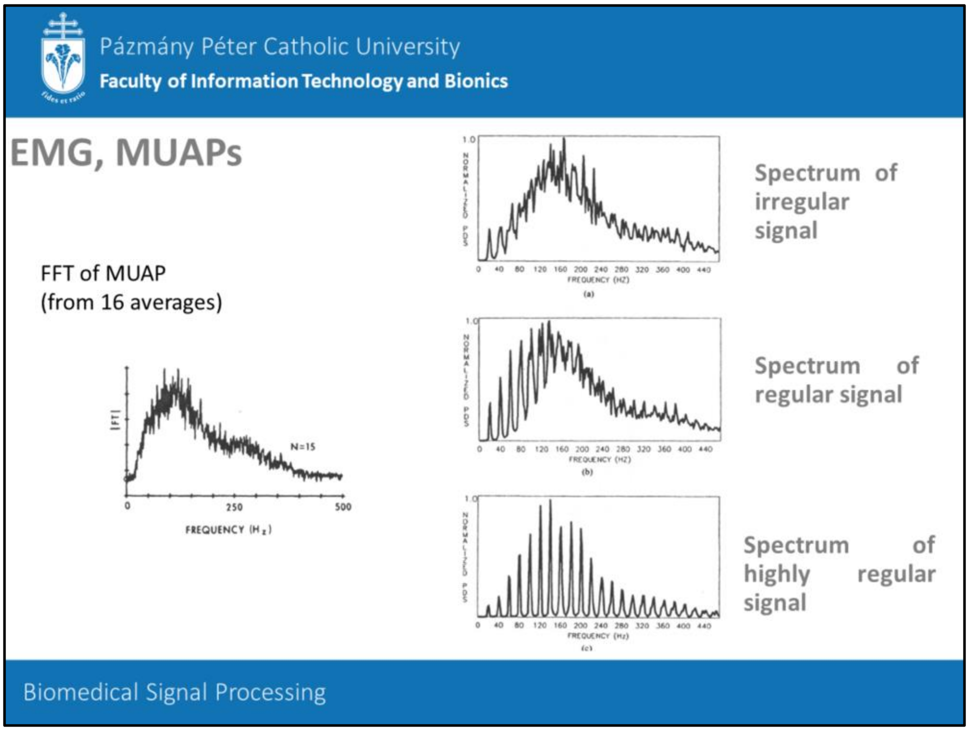


R0 is the radius of the microbubbles. Another parameter of the system is the applied resonating sound wave's pressure amplitude. A stable inertial behavior becomes progressively **more unstable as pressure increases**. (Same firing times, but different amplitudes)
The increased instability of the cavitation manifests itself as the appearance of **increased periodicity components** (period doubling, period tripling, and so on). This will cause the appearance of subharmonics – less regular spectrum and autocorrelation.

Period doubling bifurcation in a discrete dynamical system is a bifurcation in which a slight change in a parameter value in the system's equations leads to the system switching to a new behavior with twice the period of the original system (https://en.wikipedia.org/wiki/Period-doubling_bifurcation)

Biomedical signal example: similarity between **T waves** greater for every other beat
http://ieeexplore.ieee.org/abstract/document/993195/
infant breathing, **fibrilliation**: Space-Time Chaos: Characterization, Control and Synchronization
edited by S Boccaletti, H L Mancini, W González-Viñas, J Burguete, D L Valladares

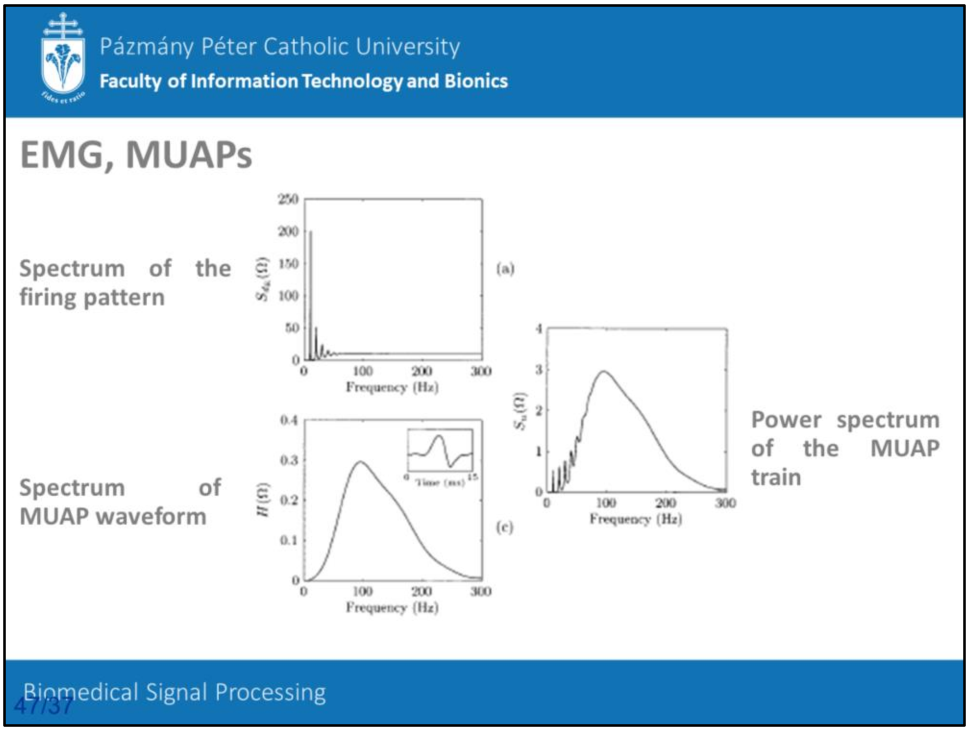
Gyöngy and Jensen (2013): Characterization of cavitation based on autocorrelation of acoustic emissions



The waveform of a MU impulse will be similar at each occurance.
If it is happening regularly in the time domain, its spectrum will also be regular (as before with cavitation, because upper harmonics appear).

simulations were carried out: what happens if waveform is regularly induced (muscle is stimulated regularly)

EMG, MUAPs(21): R p. 325, 329; *R: Rangayyan (2002): Biomedical Signal Analysis: A Case Study Approach*



EMG, MUAPs(22): SL p. 381; *SL: Sörnmo and Laguna (2005): Bioelectric Signal Processing*

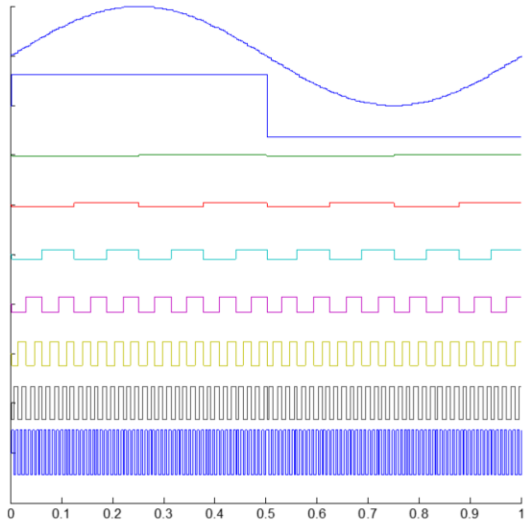
Convolution in temporal domain → multiplication of two spectra in Fourier domain. We can red from it the repetition and the spectrum of a MU wave too frequency (the small peaks incorporated into the large waveform).

EXTRA MATERIAL

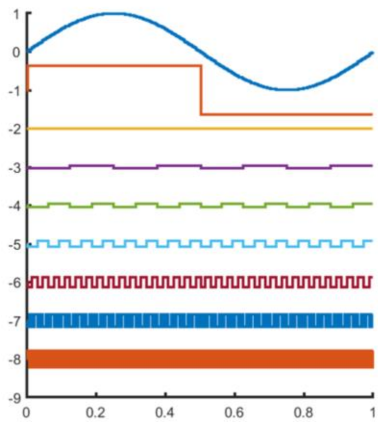
Other bases?

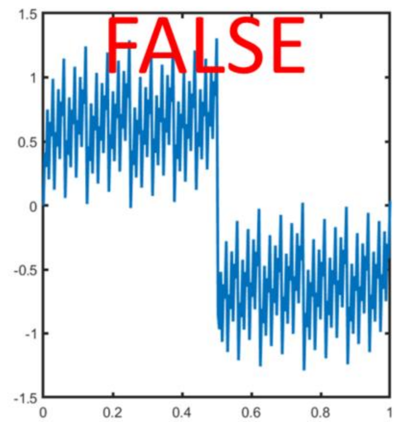
Do you know about other bases apart from the orthogonal bases of (complex) sinusoids?

Do these Haar functions sum up to the above sine?

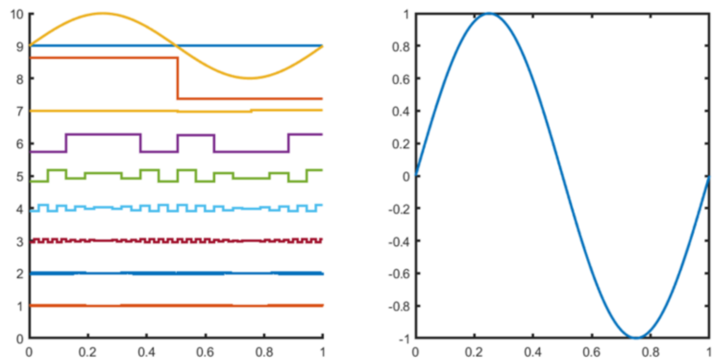


Do these Haar functions sum up to the above sine?

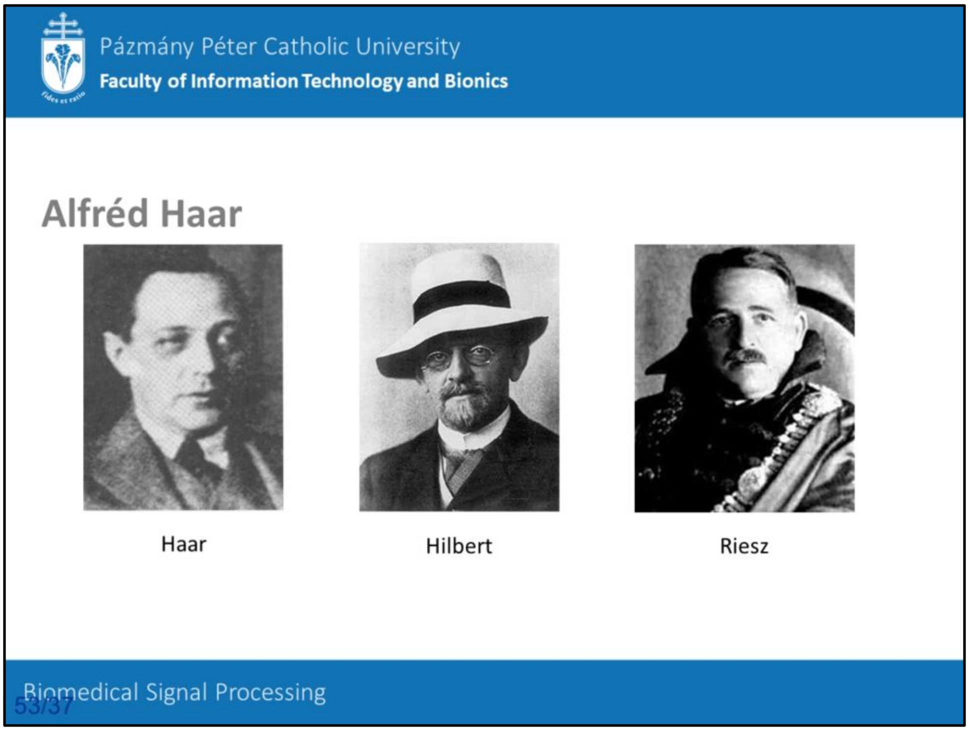




Decomposition of sines into dyadic Haar wavelets



periods so-called dyadic ($2^n, n \in P$) multiples of fundamental period form orthogonal set, but then support of basis is also local.
Will visit this later in the Wavelets lecture



b. 1885, Budapest, d. 1933, Szeged
1903: Fasori Lutheran secondary school, Budapest (notable alumni: Wigner, Harsányi, von Neumann, Teller, Kandó)
1909: doctorate, Univ. Göttingen (supervised by David Hilbert)
1912: invited with Frigyes Riesz as professors to Univ. Kolozsvár
1921: Kolozsvár becomes Cluj, Romania; Univ. Kolozsvár moves to Szeged; Haar and Riesz found its Centre of Mathematics

credit question:
Hilbert: Hilbert transform, Hilbert spaces (inner product space that is also a complete metric space)
Riesz "He had an uncommon method of giving lectures: he entered the lecture hall with an assistant and a docent. The docent then began reading the proper passages from Riesz's handbook and the assistant inscribed the appropriate equations on the blackboard—while Riesz himself stood aside, nodding occasionally."

wikipedia pages of Alfred Haar, David Hilbert, Frigyes Riesz, Fasori Gimnázium
sigma: impulse trains, random collapse t; sigma: simulations (alpha = 0 dB)

Short explanation for why sinusoids are the only orthogonal basis with integer multiple periodicities

- Express candidate basis B(t,f_0) with frequency f_0 as a complex Fourier series

$$
\begin{aligned}
B(t,f) &= \sum_{n=0}^{\infty} c_n \exp(jn2\pi f_0 t) \\
c_n &= ja_n + b_n \\
\text{If basis is not sinusoid, } \exists \quad m,n & \quad c_m, c_n \neq 0 \\
\langle B(t,mf_0)B(t,nf_0)\rangle &\neq 0
\end{aligned}
$$

(If basis is not sinusoid, in its decomposition into Fourier complex coefficients,
there will exist some pair of *m*, *n* coefficients that are non-zero.
Hence, two basis functions at these dilations will have a common component at frequency *mn*, so that the corresponding inner product will be non-zero)

Henry Scher: On Fourier series using functions other than sine and cosine

- Haar function H(t,T) and its orthogonal pair H'(t,T) are quantized equivalents of sine and cosine.

$$\frac{\pi}{4}H(t,T) = \sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots \quad (20)$$

$$\frac{\pi}{4}H'(t,T) = \cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \ldots \quad (21)$$

$$\omega = 2\pi/T \quad (22)$$

- Using result from previous slide: $\langle H(t,T)H(t,3T)\rangle \neq 0$
- However, for $n, m \in \mathbb{Z}$: $\langle H(t,2^nT)H(t,2^mT)\rangle = 0$
- This result sets the scene for dyadic signal decomposition..

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

FT of discrete signal

Let our signal $x(t)$ be sampled:

$$y(t) = \text{Ш}(f_s t)x(t),$$

where

$$\text{Ш}(t) = \sum_{n \in \mathbb{Z}} \delta(t-n)$$

Then the Fourier transform is given by:

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} \text{Ш}(f_s t)x(t)\,\mathrm{d}t \\ &= \text{Ш}(f_s T) * X(f) \end{aligned}$$

$\text{Ш}(t)$ is the comb function, it samples the signal with f_s. Its fourier transform is also a comb function, and a convolution with this comb function will periodically repeat the fourier transform of the sampled signal.