Some definitions and theorems: — *Can You sketch the accompanying Figures?*[1]

Dynamical system: Let (X,d) be a metric space and let $\mathbb{T}$ be one of the following subsets of $\mathbb{R}$: the entire real line $\mathbb{R}$, the set of integer numbers $\mathbb{Z}$, the discrete set of the form $h\mathbb{Z}$ where $h>0$ is fixed. The mapping $\Phi : \mathbb{T} \times X \to X$ is a *dynamical system on X with time* $\mathbb{T}$ if a.) Φ is continuous (jointly in the two variables) b.) $\Phi(0,x) = x$ for all $x \in X$ c.) $\Phi(t, \Phi(s,x)) = \Phi(t+s, x)$ for all $t, s \in \mathbb{T}$ and $x \in X$.

Invariant set: Let (X,d) be a metric space . The set $S \subset X$ is *invariant with respect to* the dynamical system $\Phi : \mathbb{T} \times X \to X$ if $\Phi(t,x) \in S$ for all $t \in \mathbb{T}$ and $x \in X$.[2]

Trajectory, positive half–trajectory, ω–limit set: The *trajectory through* $x \in X$ is the set $\gamma(x) = \{\Phi(t,x) \mid t \in \mathbb{T}\}$. The *positive half–trajectory through* $x \in X$ is the set $\gamma^+(x) = \{\Phi(t,x) \mid t \in \mathbb{T} \text{ and } t \geq 0\}$. The *$\omega$–limit set of the point* $x \in X$ is the set $\omega(x) = \{y \in X \mid \text{there exists a time–sequence } \{t_n\}_{n=1}^{\infty} \subset \mathbb{T} \text{ such that } t_n \to \infty \text{ and } \Phi(t_n, x) \to y\}$.

Stability, attractivity, asymptotic stability of a compact invariant set $S \subset X$: The compact invariant set $S \subset X$ is *stable* if, given $\varepsilon > 0$ arbitrarily, there exists a $\delta > 0$ such that $d(\Phi(t,x), S) < \varepsilon$ whenever $d(x,S) < \delta$ and $t \in \mathbb{T}$, $t \geq 0$.[3] The compact invariant set $S \subset X$ is *attractive* if there is an $\eta_0 > 0$ such that $d(x,S) < \eta_0$ implies that $d(\Phi(t,x), S) \to 0^+$ as $t \to \infty$ and $t \in \mathbb{T}$. The compact invariant set $S \subset X$ is *asymptotically stable* if it is both stable and attractive.[4]

Region of attraction of an asymptotically stable compact invariant set $S \subset X$: This is the (necessarily open) set $A(S) = \{x \in X \mid d(\Phi(t,x), S) \to 0^+ \text{ as } t \to \infty \text{ and } t \in \mathbb{T}\}$. [5]

Basic properties of omega–limit sets in $\mathbb{R}^d$: Let $\gamma^+(x)$ be a bounded, positive half–trajectory of the continuous–time dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Then $\omega(x)$ is a nonempty, closed, bounded and connected invariant set in $\mathbb{R}^d$. In addition, $d(\Phi(t,x), \omega(x)) \to 0^+$ as $t \to \infty$.[6]

Poincaré–Bendixson Theorem: Let $\gamma^+(x)$ be a bounded, positive half–trajectory of the continuous–time dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ and assume that Φ has only a finite number of equilibria. Then $\omega(x)$ is either an equilibrium pont x_0, or a periodic orbit Γ, or a heteroclinic cycle H.[7] In the two latter cases, the interior of Γ and the interior of H contain at least one equilibrium point.

Theorem on asymptotic stability of the origin for a linear system $\dot{x} = Ax$: The necessary and sufficient condition is that the real part of all eigenvalues of matrix A is negative. This is equivalent to the existence of a pair of positive constants $\mathcal{C}$, α such that $\|e^{At}x\| \leq \mathcal{C}\, e^{-\alpha t}\|x\|$ whenever $t \geq 0$ and $x \in \mathbb{R}^d$.[8]

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[1] They help a lot in understanding and remembering the basic features of dynamical systems.

[2] The most important examples for an invariant set are *equilibrium points* and *periodic orbits*. You should be able to formulate the definitions of stability, attractivity, asymptotic stability, and region of attraction for an equilibrium point $x_0 \in \mathbb{R}^d$ as well as for a periodic orbit $\Gamma \subset \mathbb{R}^d$. Please remember the definitions of equilibria and periodic orbits, too.

[3] The set $S \subset X$ is *compact* if, given a sequence $\{x_n\}_{n=1}^{\infty} \subset S$ arbitrarily, there exist an $x^* \in S$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x^*$ as $k \to \infty$. In short: if S is closed and every sequence in S has a convergent subsequence. A subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded. Note also that the *distance between a point $x \in X$ and a compact set $S \subset X$* is defined as $d(x,S) = \min\{d(x,y) \mid y \in S\}$.

[4] If $S \subset X$ is a compact and asymptotically stable invariant set, then S is an *attractor* and vice versa. Remember that, even for equilibria, stability and attractivity are independent notions. Are You able to recall the related counterexamples?

[5] Attractor $S \subset X$ is *global* if $A(S) = X$.

[6] The standard example for a continuous–time dynamical system in $\mathbb{R}^d$ is the solution operator of an autonomous ordinary differential equation with the properties of global existence (i.e., existence for all $t \in \mathbb{R}$), uniqueness, and continuous dependence on initial conditions.

[7] You should be able to formulate the definition of a heteroclinic cycle.

[8] Thus asymptotic stability for a (constant coefficient) linear system is equivalent to *exponential stability*.

One–step p–th order ($p \in \mathbb{N}$, $p \geq 1$), stepsize h ($0 < h \leq h_0$]) discretization operator for equation $\dot{x} = f(x)$ inducing a C^{p+1} dynamical system Φ on $\mathbb{R}^d$: A $C^{p+1} = C^{p+1}([0,h_0] \times \mathbb{R}^d, \mathbb{R}^d)$ mapping $\phi : [0,h_0] \times \mathbb{R}^d \to \mathbb{R}^d$ is a *one–step p–th order ($p \in \mathbb{N}$, $p \geq 1$), stepsize h ($0 < h \leq h_0$) discretization operator for equation $\dot{x} = f(x)$* if a.) for constant $K > 0$ suitably chosen, $\|\Phi(h,x) - \phi(h,x)\| \leq Kh^{p+1}$ whenever $0 \leq h \leq h_0$ and $x \in \mathbb{R}^d$ b.) for stepsize h small, $\phi(h,x)$ can be effectively computed on the basis of knowing the behaviour of function f near $x \in \mathbb{R}^d$.[9]
Grobman–Hartman Lemma: Consider the differential equation $\dot{x} = f(x)$ where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a continuously differentiable function, $f(0) = 0 \in \mathbb{R}^d$ and $f'(0) = A \in L(\mathbb{R}^d, \mathbb{R}^d)$, a $d \times d$ matrix with eigenvalues λ_k, $k = 1, 2, \ldots, d$. Assume that $\operatorname{Re}\lambda_k \neq 0$ for each k. Then, in a small neighborhood of the origin $0 \in \mathbb{R}^d$, the nonlinear equation (N) $\dot{x} = f(x)$ with solution operator $\Phi(t,x)$, the linearized equation (L) $\dot{x} = Ax$ with solution operator $e^{tA}x$, and—for stepsize h sufficiently small—the discretized equation (D) $X = \phi(h,x)$ with solution operator $\phi(h,x)$ are essentially the same. Loosely speaking, in a small neighborhood of a nondegenerate equilibrium, both linearization and discretization are almost–identity coordinate transformations that, preserving time, map trajectory segments into trajectory segments. Stable and unstable local manifolds/subspaces of the origin are mapped to each other and they are tangent at the origin to each other.[10]
Periodic orbits of Lotka–Volterra systems $\dot{x} = x(c_1 + a_1 x + b_1 y)$, $\dot{y} = y(c_2 + a_2 x + b_2 y)$: There is only one possibility for a Lotka–Volterra system to have periodic orbits in the positive quadrant: if the positive quadrant is filled by periodic orbits, encircling about the same equilibrium point (which is a center).
The derivative of C^1 function $V : \mathcal{N} \to \mathbb{R}$ along the trajectories of a local dynamical system $\Phi(t,x)$ induced by the autonomous differential equation (E) $\dot{x} = f(x)$ where $f : \mathcal{N} \to \mathbb{R}^d$ is a C^1 (in words: a continuously differentiable) function and the related consequences: The above–mentioned derivative is simply

$$\dot{V}_{(E)}(x) = \frac{d}{dt} V(\Phi(t,x))\big|_{t=0} = \langle \underline{grad} V(x), f(x) \rangle \quad \text{for each} \quad x \in \mathcal{N}$$

where $\mathcal{N} \subset \mathbb{R}^d$ is open. Inequalities for $\dot{V}_{(E)}(x)$ imply various consequences on stability properties of x_0 as follows: Nested level surfaces around an equilibrium point $x_0 \in \mathcal{N}$ which is a local minimum of function V and the sharp inequality $\dot{V}_{(E)}(x) < 0$ on the set $\mathcal{N} \setminus \{x_0\}$ imply that x_0 is asymptotically stable. If only $\dot{V}_{(E)}(x) \leq 0$ on the set $\mathcal{N} \setminus \{x_0\}$, then x_0 is stable. Reformulations on instability properties are at hand.

[9] You should be able to define at least both the explicit and the implicit Euler method as well as to recall the contraction mapping principle the definition of the implicit Euler method is based on. In order to define the unstable invariant manifold of the origin with respect to the discretized dynamics, it should be mentioned that $\phi(h,\cdot)$ is an invertible function of class C^{p+1}.

[10] The precise technical statement is that there exist a neighborhood $\mathcal{U}$ of the origin $0 \in \mathbb{R}^d$, a homeomorphism $\mathcal{H} : \mathcal{U} \to \mathcal{H}(\mathcal{U})$ and, for h_0 sufficiently small, a one–parameter family of homeomorphisms $\mathcal{H}_h : \mathcal{U} \to \mathcal{H}_h(\mathcal{U})$, $h \in (0,h_0]$ with the properties that $\mathcal{H}(0) = \mathcal{H}_h(0) = 0$ and, as long as the trajectory segments remain in $\mathcal{U}$, $\mathcal{H}(\Phi(t,x)) = e^{At}\mathcal{H}(x)$ and $\mathcal{H}_h(\Phi(h,x)) = \phi(h, \mathcal{H}_h(x))$. Moreover, $\mathcal{H}$ and $\mathcal{H}_h$ can be chosen in such a way that they are differentiable at 0 and satisfy $\mathcal{H}'(0) = \mathcal{H}'_h(0) = \mathrm{id}_{\mathbb{R}^d}$. In addition, there exists a constant $K > 0$ such that $\|\mathcal{H}_h(x) - x\| \leq Kh^p$ for each $h \in (0,h_0]$, $x \in \mathcal{U}$. In the coordinate system $\mathcal{Y} \times \mathcal{Z} = \mathbb{R}^d$ near the origin, the local unstable manifolds for $\Phi(t,\cdot)$ and $\phi(h,\cdot)$ can be represented as the graphs of the locally defined C^{p+1} functions $u, u_h : \mathcal{Y} \to \mathcal{Z}$ where $u(0_\mathcal{Y}) = u_h(0_\mathcal{Y}) = 0_\mathcal{Z}$ and $u'(0_\mathcal{Y}) = u'_h(0_\mathcal{Y}) = 0 \in L(\mathcal{Y}, \mathcal{Z})$ and $\|u(y) - u_h(y)\| \leq \text{const}\, h^p$. Here $\mathcal{Y}$ and $\mathcal{Z}$ are linear subspaces spanned by the generalized eigenspaces belonging to eigenvalues λ_k with $\operatorname{Re}\lambda_k > 0$ and $\operatorname{Re}\lambda_k < 0$, respectively. You should be able to define stable and unstable manifolds near nondegenerate equilibria of (N), (L), (D).

Deterministic chaos: (*an informal definition*) Complexity of the discrete–time or continuous–time dynamical systems from the view–points of *topology* (sensitive dependence on initial conditions), *measure theory* (density functions for the asymptotic behavior of the trajectories, connections between time and space averages), and *combinatorics* (an uncountable choice of coding sequences like those with alphabet[11] L, R addressing consecutive points on certain trajectories).

Devaney's Definition of Chaos for Dynamical Systems with Time $\mathbb{T} = \mathbb{N}$: (*one of the competing formal definitions*)[12]: Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous function. The dynamics generated by the iterates of f is chaotic if properties a.) *sensitive dependence on initial conditions*, b.) *periodic orbits with long periods are dense*, c.) *there is a dense orbit* hold true:

$$\begin{aligned}
&a.)\ \exists\, \eta_0\ \forall\, \varepsilon\ \forall\, x\ \exists\, N\ \exists\, \tilde{x}\ \text{ such that }\ d(\tilde{x}, x) < \varepsilon\ \text{ and }\ d(f^N(\tilde{x}), f^N(x)) > \eta_0\ ,\\
&b.)\ \forall\, \varepsilon\ \forall\, x\ \forall\, N\ \exists\, \tilde{x}\ \exists\, \tilde{N}\ \text{ such that }\ d(\tilde{x}, x) < \varepsilon\ ,\ \tilde{N} > N \text{ and }\ f^{\tilde{N}}(\tilde{x}) = \tilde{x}\ ,\\
&c.)\ \exists\, x^*\ \forall\, \varepsilon\ \forall\, x\ \exists\, N\ \text{ such that }\ d(f^N(x^*), x) < \varepsilon\ .
\end{aligned}$$

Time averages and space averages of the logistic map $F : [0, 1] \to [0, 1]$, $x \to 4x(1 - x)$: There exists an exceptional set $E \subset [0, 1]$ of measure zero such that the recursion $x_{n+1} = F(x_n)$, $n = 0, 1, \ldots$ satisfies

$$\lim_{N\to\infty} \frac{\#\{0 \le n \le N \mid x_n \in [a, b]\}}{N + 1} = \int_a^b \rho(x)\, dx \qquad \forall\ x_0 \in [0, 1] \setminus E\ \ \forall\ [a, b] \subset [0, 1]$$

where $\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$, $x \in (0, 1)$ is a *density function*.[13]

Period Three Implies Chaos Theorem: Let $f : [a, b] \to [a, b]$ be a continuous function and assume f admits a period–three orbit $x_3 = f(x_2)$, $x_2 = f(x_1)$, $x_1 = f(x_3)$ with minimal period 3. Then, at least on a closed subset of $[a, b]$, f is chaotic.

More precisely, there exists a pair of disjoint, closed intervals $L, R \subset [a, b]$ such that $F^2(L) \supset L \cup R$ and $F^2(R) \supset L \cup R$. Thus the transition graph[14] $\mathcal{G}$ of $\phi = F^2$ has two vertices L, R, and four directed edges $L \to L$, $L \to R$, $R \to L$, $R \to R$ (both the first and the last edge are loop edges).

[11] typically, letters L ("left") and R ("right") carry a geometrical meaning

[12] The formal mathematical definitions contain a number of conditions fulfilled by some standard examples (the logistic map with $a = 4$, Lorenz attractor and Chua circuit for the usual parameters) for chaos. Physicists speak about chaos if the maximum Lyapunov exponent is positive. For C^1 mappings $f : [a, b] \to [a, b]$ there is only one Lyapunov exponent, namely

$$\lambda_{Ljap}(x_0) = \limsup_{N\to\infty} \frac{1}{N} \ln |f'(f^{N-1}(x_0)) \cdot f'(f^{N-2}(x_0)) \cdot \ldots \cdot f'(f(x_0)) \cdot f'(x_0)|\ . \tag{1}$$

Geometrically, $\lambda_{Ljap}(x_0)$ measures the exponential rate at which errors grow. In cases relevant to physics, the limes superior in (1) is, up to a set of initial points of measure zero, a limit and it does not depend on the initial point x_0. For mathematicians, *the positivity of the maximum Lyapunov exponent is just an indicator for chaos.* Mathematicians prefer Devaney's definition for chaos (which can be easily modified to the case $\mathbb{T} \supset \mathbb{N}$. For the logistic map $F : [0, 1] \to [0, 1]$, $F(x) = ax(1 - x)$ with $a = 4$, the Lyapunov exponent is $\ln(2)$.)

[13] By definition, $\rho(x) \ge 0$ and $\int_0^1 \rho(x)\, dx = 1$.

[14] In general, if $I_1, I_2, \ldots, I_N$ are disjoint and closed subintervals of a closed and bounded interval I, and $\phi : I \to I$ is a continuous mapping, then $I_i \to I_j$ is a directed edge of *the transition graph* $\mathcal{G}$ with vertex set $I_1, I_2, \ldots, I_N$ if and only if $f(I_i) \supset I_j$. The sufficient condition for chaos in this generalized setting is that $\mathcal{G}$ has two oriented circles with nonempty intersection.

Let $\{Q_k\}_{k\in\mathbb{Z}}$ be a doubly–infinite L–R sequence. Then there exists a doubly–infinite sequence $\{x_k\}_{k\in\mathbb{Z}}$ of points in $L\cup R$ such that $x_k \in Q_k$ and $x_{k+1} = \phi(x_k)$ for each $k \in \mathbb{Z}$. In other words, there exists a doubly–infinite trajectory of ϕ visiting intervals L and R in the prescribed order. This means that *symbolic itineraries in the transition graph $\mathcal{G}$ can be represented by genuine trajectories.*

Box dimension of bounded subsets of $\mathbb{R}^d$: Let $\mathcal{C}_\varepsilon$ be the usual grid partition of $\mathbb{R}^d$ by d–dimensional cubes of side length $\varepsilon > 0$. Individual cubes are denoted by C. Consider a bounded subset A of $\mathbb{R}^d$ and set $N(\varepsilon) = \#\{C \in \mathcal{C}_\varepsilon \mid C \cap A \neq \emptyset\}$. The *upper and the lower box dimensions of* A are defined by

$$\dim_B^-(A) = \liminf_{\varepsilon\to 0^+} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)} \quad \text{and} \quad \dim_B^+(A) = \limsup_{\varepsilon\to 0^+} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)} \,,$$

respectively. In case the upper and the lower box dimensions of A coincide, we say that *the box dimension* of the set A is defined[15] and is denoted by $\dim_B(A)$. Property $\dim_B(A) \notin \mathbb{N}$ is an important fractal indicator.[16] Sets with noninteger box dimension can be termed fractals but the majority of the competing definitions requires some type of self–similarity as well. This is definitely the case for the well–known examples listed in the previous footnote.

Borel's Normal Number Theorem: There exists an exceptional set $E \subset [0,1]$ of measure zero such that every $x \in [0,1] \setminus E$ is a *normal number*. Given $\beta = 2, 3, \ldots$ arbitrarily, consider the representation

$$x = \sum_{n=1}^{\infty} \frac{j_n(x)}{\beta^n} \quad , \qquad j_n(x) \in \{0\,,\,1\,,\,\ldots\,,\beta-1\} \tag{2}$$

of $x \in [0,1]\setminus E$ in the number system with base β. Then for all integer $K = 1, 2, \ldots$, the relative frequency of every string $s_1 s_2 \ldots s_K$ (where $s_k \in \{0, 1, \ldots, \beta-1\}$ for each $k = 1, 2, \ldots, K$) of length K in (2) is independent of the choice of the string and equals to β^{-K}. Thus finite sequences in every base β are distributed uniformly.[17]

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[15] This is in fact a generalization of the classical concept of the dimension. For example, the d dimensional cube $A = [0, N_0]^d$ of side length $N_0 \in \mathbb{N}$ is covered by $(N_0 k)^d$ cells of the grid $\mathcal{C}_{1/k}$ implying

$$\dim_B([0,N_0]^d) = \lim_{k\to\infty} \frac{\ln((N_0 k)^d)}{\ln(k)} = \lim_{k\to\infty} \frac{d(\ln(N_0) + \ln(k))}{\ln(k)} = d\,.$$

Observe that the box dimension of a bounded set $A \subset \mathbb{R}^d$ with nonempty interior is d.

[16] For example,

$$\dim_B(\text{Sierpinski triangle}) = \frac{\ln(3)}{\ln(2)}\,,\ \dim_B(\text{Koch curve}) = \frac{\ln(4)}{\ln(3)}\,,\ \dim_B(\text{Cantor set}) = \frac{\ln(2)}{\ln(3)}\,,\ \dim_B(\text{Barnsley fern}) \approx 1.45\ldots.$$

Explicit formulas for $\dim_B$(Barnsley fern) are not known – hence the last result is due to computer experimentation. Construction and (a somewhat heuristic derivation of the value of the) box dimension in the first three examples are a must. The standard example for *chaos game* is based on *random iterations of the constitutive affine contractions* leading to the construction of the Sierpinski triange. Convergence of the random iterations is guaranteed by the theorem below.

[17] To be a string $s_1 s_2 \ldots s_K$ of length K in (2) means that $s_k = j_{n^*+k}(x)$ for some $n^* \in \{1, 2, \ldots\}$ and each $k = 1, 2, \ldots, K$. In particular,

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N \mid j_n(x) = \ell\}}{N} = \frac{1}{\beta} \qquad \forall\, \ell \in \{0\,,\,1\,,\,\ldots\,,\beta-1\}\,.$$

The same uniformity is valid for all possible pairs, triplets etc. of digits. No digit or string is "favored". We are facing an asymptotic statement on time averages and space averages in consecutive finite structures with constant density functions.