

# Picard-Lindelöf Theorem

Lecture 4  
Math 634  
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**Theorem** *The space  $\mathcal{C}([a, b])$  of continuous functions from  $[a, b]$  to  $\mathbb{R}^n$  equipped with the norm*

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in [a, b]\}$$

*is a Banach space.*

**Definition** Two different norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on a vector space  $\mathcal{X}$  are *equivalent* if there exist constants  $m, M > 0$  such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

for every  $x \in \mathcal{X}$ .

**Theorem** *If  $(\mathcal{X}, \|\cdot\|_1)$  is a Banach space and  $\|\cdot\|_2$  is equivalent to  $\|\cdot\|_1$  on  $\mathcal{X}$ , then  $(\mathcal{X}, \|\cdot\|_2)$  is a Banach space.*

**Theorem** *A closed subspace of a complete metric space is a complete metric space.*

We are now in a position to state and prove the Picard-Lindelöf Existence-Uniqueness Theorem. Recall that we are dealing with the IVP

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = a. \end{cases} \quad (1)$$

**Theorem (Picard-Lindelöf)** *Suppose  $f : [t_0 - \alpha, t_0 + \alpha] \times \overline{\mathcal{B}(a, \beta)} \rightarrow \mathbb{R}^n$  is continuous and bounded by  $M$ . Suppose, furthermore, that  $f(t, \cdot)$  is Lipschitz continuous with Lipschitz constant  $L$  for every  $t \in [t_0 - \alpha, t_0 + \alpha]$ . Then (1) has a unique solution defined on  $[t_0 - b, t_0 + b]$ , where  $b = \min\{\alpha, \beta/M\}$ .*

*Proof.* Let  $\mathcal{X}$  be the set of continuous functions from  $[t_0 - b, t_0 + b]$  to  $\overline{\mathcal{B}(a, \beta)}$ . The norm

$$\|g\|_w := \sup\{e^{-2L|t-t_0|}|g(t)| \mid t \in [t_0 - b, t_0 + b]\}$$

is equivalent to the standard supremum norm  $\|\cdot\|_\infty$  on  $\mathcal{C}([t_0 - b, t_0 + b])$ , so this vector space is complete under this weighted norm. The set  $\mathcal{X}$  endowed with this norm/metric is a closed subset of this complete Banach space, so  $\mathcal{X}$  equipped with the metric  $d(x_1, x_2) := \|x_1 - x_2\|_w$  is a complete metric space.

Given  $x \in \mathcal{X}$ , define  $T(x)$  to be the function on  $[t_0 - b, t_0 + b]$  given by the formula

$$T(x)(t) = a + \int_{t_0}^t f(s, x(s)) \, ds.$$

Step 1: If  $x \in \mathcal{X}$  then  $T(x)$  makes sense.  
This should be obvious.

Step 2: If  $x \in \mathcal{X}$  then  $T(x) \in \mathcal{X}$ .  
If  $x \in \mathcal{X}$ , then it is clear that  $T(x)$  is continuous (and, in fact, differentiable). Furthermore, for  $t \in [t_0 - b, t_0 + b]$

$$|T(x)(t) - a| = \left| \int_{t_0}^t f(s, x(s)) \, ds \right| \leq \left| \int_{t_0}^t |f(s, x(s))| \, ds \right| \leq Mb \leq \beta,$$

so  $T(x)(t) \in \overline{\mathcal{B}(a, \beta)}$ . Hence,  $T(x) \in \mathcal{X}$ .

Step 3:  $T$  is a contraction on  $\mathcal{X}$ .  
Let  $x, y \in \mathcal{X}$ , and note that  $\|T(x) - T(y)\|_w$  is

$$\sup \left\{ e^{-2L|t-t_0|} \left| \int_{t_0}^t [f(s, x(s)) - f(s, y(s))] \, ds \right| \mid t \in [t_0 - b, t_0 + b] \right\}.$$

For a fixed  $t \in [t_0 - b, t_0 + b]$ ,

$$\begin{aligned} & e^{-2L|t-t_0|} \left| \int_{t_0}^t [f(s, x(s)) - f(s, y(s))] \, ds \right| \\ & \leq e^{-2L|t-t_0|} \left| \int_{t_0}^t |f(s, x(s)) - f(s, y(s))| \, ds \right| \\ & \leq e^{-2L|t-t_0|} \left| \int_{t_0}^t L|x(s) - y(s)| \, ds \right| \\ & \leq Le^{-2L|t-t_0|} \left| \int_{t_0}^t \|x - y\|_w e^{2L|s-t_0|} \, ds \right| \\ & = \frac{\|x - y\|_w}{2} (1 - e^{-2L|t-t_0|}) \\ & \leq \frac{1}{2} \|x - y\|_w. \end{aligned}$$

Taking the supremum over all  $t \in [t_0 - b, t_0 + b]$ , we find that  $T$  is a contraction (with  $\lambda = 1/2$ ).

By the contraction mapping principle, we therefore know that  $T$  has a unique fixed point in  $\mathcal{X}$ . This means that (1) has a unique solution in  $\mathcal{X}$  (which is the only place a solution could be).  $\square$