Czufiz V 2

Térbeli sajátérték-probléma:

$$\Delta A_n(r) = -k_n^2 A_n(r)$$

→ Probléma: $\dfrac{d^2 A_n(x)}{dx^2} = -k_n^2 A_n(x)$

↳ $A_n(x) = c_1 \sin(k_n x) + c_2 \cos(k_n x)$

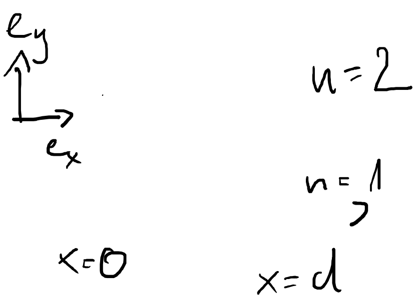
→ Elektromos mező

$$E_n(\underline{r};t) = -\frac{d f_n(t)}{dt} \cdot A_n(\underline{r})$$

a mező iránya e_y

és $A_n(0) = A_n(d) = 0$

$$kd = n\pi \rightarrow k_n = \frac{n\pi}{d};\ n = 1;..$$



Magyarázat:

① $\Delta A(\underline{r};t) - \dfrac{1}{c^2} \dfrac{\partial A^2(\underline{r};t)}{\partial t^2} = 0$

② $B(\underline{r};t) = \nabla \times A(\underline{r};t)$

③ $E(\underline{r},t) = -\partial A(\underline{r},t)$

③ $$E(\underline{r},t) = -\frac{\partial A(\underline{r},t)}{\partial t}$$

④ $$A(\underline{r},t) = f(t)\cdot A(\underline{r})$$

①+④ $$f(t)\,\Delta A(\underline{r}) = A(\underline{r})\,\frac{1}{c^2}\,\frac{d^2 f(t)}{dt^2}$$

$$\hookrightarrow \frac{\Delta A(\underline{r})}{A(\underline{r})} = \frac{\frac{1}{c^2}\frac{d f^2}{dt^2}}{f(t)} = -k^2$$

$$\rightarrow \Delta A_k(\underline{r}) = -k^2\, A_k(\underline{r})$$

Def: $\omega_k = kc$

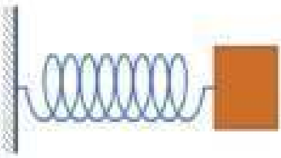
$$\frac{d^2 f_k(t)}{dt^2} = -k^2 c^2 f_k(t) = -\omega_k^2 f_k(t)$$

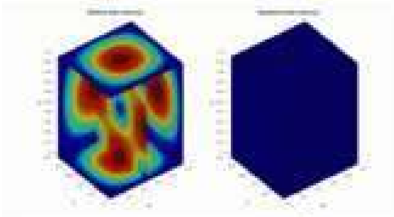
⇓

Cavity modes sajátértékproblémájának megoldása

$$\left(A(\underline{r},t) = \sum_n f_n(t)\,A_n(\underline{r})\right)$$

$$\left(\int_V \left(\nabla \times A_k(\underline{r})\right)^2 dV = k^2\right)$$





$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Cx^2 \qquad L_k = \frac{1}{2}\varepsilon_0 \dot{f}_k^2 - \frac{k^2}{2\mu_0} f_k^2$$

$$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Cx^2 \qquad H_k = \frac{1}{2}\varepsilon_0 \dot{f}_k^2 + \frac{k^2}{2\mu_0} f_k^2$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad E_{kn} = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$n = 0,1,2,\ldots,n,\ldots$$

$$\omega = \sqrt{\frac{C}{m}}, \qquad \omega_{mnp} = \pi \cdot c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{L^2}}$$

Energy of an Electromagnetic Cavity Mode

At every resonant frequeny $\omega_{m\ell p}$ the energy is „quantized'

$$E_n^{m\ell p} = \hbar\omega_{m\ell p} \cdot \left(n + \frac{1}{2}\right) \qquad n = 0,1,2,\ldots,n,\ldots$$

The energy of a photon with frequency $\omega_{m\ell p}$ is $\hbar\,\omega_{m\ell p}$

$E_0 = \frac{1}{2}\hbar\omega$? $\quad E_1 = E_0 + \hbar\omega \quad E_2 = E_0 + 2\hbar\omega \quad E_n = E_0 + n\hbar\,\omega$

? + One photon + 2 photons + n photons

$$E_0 \qquad \sum_{mlp} E_0^{\omega_{m\ell p}} = \sum_{mlp} \frac{1}{2}\hbar\,\omega_{m\ell p} \to \infty \qquad 0 < \omega_{m\ell p} < \infty$$

The Hilbert Space of Quantum Mechanics

Information we can know about a state

In quantum physics a ***physical state*** is represented by a ***state vector*** in a complex vector space, called ***Hilbert space***.

We call the state vector a ***„ket"***, and denote it by $|\alpha\rangle$

(This state ket is postulated to contain complete information about the physical state. Everything we are allowed to ask about the state is contained in the ket.)

Two kets can be added, and can be multiplied by a complex number, and the result is also a ket

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle \qquad \lambda|\alpha\rangle = |\alpha\rangle\lambda$$

Null ket $\lambda|\alpha\rangle$ if $\lambda = 0$

$|\alpha\rangle$ and $\lambda|\alpha\rangle$, with $\lambda \neq 0$, represent the same physical state.

(Only the „direction" in vector space is of significance. We are dealing with rays rather than vectors .)

The state space of quantum physics is a Hilbert space

$$|a\rangle, |b\rangle \in \mathscr{H} \quad |a\rangle + |b\rangle = |c\rangle \in \mathscr{H}; \quad \lambda|a\rangle \in \mathscr{H} \quad \lambda \text{ complex number}$$

The dimension of the vector space depends on the physical system (Spin : 2; Finite dim.: *n*; Bounded: countable infinite; Free: continuously infinite)

Observable is represented by a linear operator

The operator acts on the ket from left, and maps a ket on a ket.

$$\hat{\mathbf{A}} \cdot (|a\rangle) = \hat{\mathbf{A}}|a\rangle \in \mathscr{H}$$

The Hilbert space is a linear vector space over the complex numbers, in which the scalar product of the elements exist.

Definition of the scalar product of a bra and a ket

$\langle b|a\rangle$ = number (in general complex), for which $\langle b|a\rangle = \langle a|b\rangle^*$

Scalar product is a „**bra**c**ket**"

$\langle a|a\rangle$ = real number; $\langle a|a\rangle \geq 0$; If $\langle a|a\rangle = 0 \rightarrow |a\rangle$ null ket

To kets are orthogonal

$$|a\rangle \perp |b\rangle \quad \text{if} \quad \langle a|b\rangle = 0$$

Normalized ket:

$$|\bar{a}\rangle = \frac{1}{\sqrt{\langle a|a\rangle}}|a\rangle \quad \rightarrow \langle \bar{a}|\bar{a}\rangle = 1$$

Bra	Ket	Bra-ket	Operator
$\langle b\|$	$\|a\rangle$	$\langle b\|a\rangle$	$\hat{\mathbf{A}}$

Operators $\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}, \ldots \quad \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \ldots$

$$\hat{\mathbf{X}} = \hat{\mathbf{Y}} \quad \text{if} \quad \hat{\mathbf{X}}|a\rangle = \hat{\mathbf{Y}}|a\rangle \quad \text{for } \forall|a\rangle$$

$$\text{null operator if } \hat{\mathbf{X}}|a\rangle = 0 \quad \text{for } \forall|a\rangle$$

$$\hat{\mathbf{X}} + \hat{\mathbf{Y}} = \hat{\mathbf{Y}} + \hat{\mathbf{X}}; \quad \hat{\mathbf{X}} + \left(\hat{\mathbf{Y}} + \hat{\mathbf{Z}}\right) = \left(\hat{\mathbf{X}} + \hat{\mathbf{Y}}\right) + \hat{\mathbf{Z}}$$

Linear Operators

$$\hat{\mathbf{X}}\left(c_a|a\rangle + c_b|b\rangle\right) = c_a\hat{\mathbf{X}}|a\rangle + c_b\hat{\mathbf{X}}|b\rangle$$

Matrix representation of kets bras and operators

In quantum physics the mathematical representation of observables are linear self-adjoint operators

Lemma 1. Eigenvalues of self-adjoint operators are real numbers. The eigen-kets belonging to different eigenvalues are orthogonal.

$\hat{\mathbf{A}}|n\rangle = a^{(n)} \cdot |n\rangle \qquad a^{(1)}, a^{(2)}, \ldots, a^{(n)}, \ldots \text{eigenvalues of } \hat{\mathbf{A}}$

$|1\rangle, |2\rangle, \ldots, |n\rangle, \ldots \text{"eigen-kets"}$

$$\langle i | j \rangle = \delta_{ij}$$

Lemma 2. Eigen-kets constitute a complete orthonormal basis.

$$|a\rangle = \sum_n c^{(n)} |n\rangle \qquad c^{(n)} = \langle n | a \rangle$$

$$|a\rangle = \sum_n |n\rangle\langle n | a\rangle \qquad \sum_n |n\rangle\langle n| = \mathbf{1}$$

$$\langle a | a \rangle = \left\langle a \left| \sum_n |n\rangle\langle n| \right| a \right\rangle = \sum_n |\langle n | a\rangle|^2$$

If $|a\rangle$ is normalized, then $\sum_n |c_n|^2 = \sum_n |\langle n | a \rangle|^2 = 1$

Projection operator $\hat{\Lambda}_n = |n\rangle\langle n|$

$$\hat{\Lambda}_{a^{(n)}} |a\rangle = |n\rangle\langle n | a\rangle = c_n |n\rangle \qquad \sum_n \hat{\Lambda}_n = \hat{1} \quad \text{(Completeness)}$$

If N is the dimension of the space of kets, the representation of operator **X**

$$\hat{\mathbf{X}} = \hat{\mathbf{X}} \to \hat{\mathbf{X}} = \left(\sum_{n=1}^{N} |n\rangle\langle n|\right) \hat{\mathbf{X}} \left(\sum_{m=1}^{N} |m\rangle\langle m|\right) =$$

$$= \sum_{m=1}^{N} \sum_{n=1}^{N} |m\rangle\langle m|\mathbf{X}|n\rangle\langle n|$$

$\langle m|\mathbf{X}|n\rangle$ N^2 number; $\langle m|$ is a row vector;

$|n\rangle$ is a column vector

In order to simplify the mathematical description of the cavity dynamics, we introduce two new operators.

$\hat{\mathbf{a}}^\dagger$ The *"creation" or "raising" operator* will increase the number of photons in the cavity by one,

$\hat{\mathbf{a}}$ the *"annihilation" or "lowering"* operator will decrease the number of photons by one.

First, we introduce these new operators for the simple quantum mechanical harmonic oscillator.

$\hat{\mathbf{x}}$	Position operator	$\hat{\mathbf{x}} = x\cdot$
$\hat{\mathbf{p}}$	Momentum operator	$\hat{\mathbf{p}} = -\mathrm{j}\hbar\frac{\partial}{\partial x}$

Creation and Annihilation Operators of a Harmonic Oscillator

The time-independent Schrödinger equation for the harmonic oscillator

$$\hat{\mathbf{H}}|\psi\rangle = E|\psi\rangle; \quad \hat{\mathbf{H}} = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2}m\omega^2\hat{\mathbf{x}}^2 \; = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2$$

Let us define a (non-Hermitian) operator

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{\mathbf{p}} - \mathrm{j}m\omega\hat{\mathbf{x}}\right)$$

and its adjoint

$$\hat{\mathbf{a}}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{\mathbf{p}} + \mathrm{j}m\omega\hat{\mathbf{x}}\right)$$

We can express the position and momentum operators by the new operators as

$$\hat{\mathbf{x}} = \mathrm{j}\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{\mathbf{a}} - \hat{\mathbf{a}}^\dagger\right) \qquad \hat{\mathbf{p}} = \sqrt{\frac{m\hbar\omega}{2}}\left(\hat{\mathbf{a}} + \hat{\mathbf{a}}^\dagger\right)$$

We know that the commutator $\left[\hat{\mathbf{x}},\hat{\mathbf{p}}\right] \equiv \hat{\mathbf{x}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{x}} = \mathrm{j}\hbar$

The commutator $\left[\hat{\mathbf{a}},\hat{\mathbf{a}}^\dagger\right] \equiv \hat{\mathbf{a}}\hat{\mathbf{a}}^\dagger - \hat{\mathbf{a}}^\dagger\hat{\mathbf{a}} = \hat{\mathbf{1}}$

Proof

$$\left[\hat{\mathbf{a}},\hat{\mathbf{a}}^\dagger\right] \equiv \hat{\mathbf{a}}\hat{\mathbf{a}}^\dagger - \hat{\mathbf{a}}^\dagger\hat{\mathbf{a}} = \frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{\mathbf{p}} - \mathrm{j}m\omega\hat{\mathbf{x}}\right)\frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{\mathbf{p}} + \mathrm{j}m\omega\hat{\mathbf{x}}\right) -$$

$$-\frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{\mathbf{p}} + \mathrm{j}m\omega\hat{\mathbf{x}}\right)\frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{\mathbf{p}} - \mathrm{j}m\omega\hat{\mathbf{x}}\right) =$$

$$= \frac{1}{2m\hbar\omega}\left[\hat{\mathbf{p}}^2 + \mathrm{j}m\omega\left(\hat{\mathbf{p}}\hat{\mathbf{x}} - \hat{\mathbf{x}}\hat{\mathbf{p}}\right) + m^2\omega^2\hat{\mathbf{x}}^2\right] -$$

$$-\frac{1}{2m\hbar\omega}\left[\hat{\mathbf{p}}^2 - \mathrm{j}m\omega\left(\hat{\mathbf{p}}\hat{\mathbf{x}} - \hat{\mathbf{x}}\hat{\mathbf{p}}\right) + m^2\omega^2\hat{\mathbf{x}}^2\right] \quad = \frac{\mathrm{j}}{\hbar}\left(\hat{\mathbf{p}}\hat{\mathbf{x}} - \hat{\mathbf{x}}\hat{\mathbf{p}}\right) = -\frac{\mathrm{j}}{\hbar}\mathrm{j}\hbar\hat{\mathbf{1}} = \hat{\mathbf{1}}$$

The Hamilton operator

$$\hat{\mathbf{H}} = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2}m\omega^2\hat{\mathbf{x}}^2 = \frac{1}{2m}\frac{m\hbar\omega}{2}\left(\hat{\mathbf{a}} + \hat{\mathbf{a}}^\dagger\right)^2 - \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}\left(\hat{\mathbf{a}} - \hat{\mathbf{a}}^\dagger\right)^2 =$$

$$= \frac{\hbar\omega}{4}\left(\hat{\mathbf{a}} + \hat{\mathbf{a}}^\dagger\right)^2 - \frac{\hbar\omega}{4}\left(\hat{\mathbf{a}} - \hat{\mathbf{a}}^\dagger\right)^2 = \frac{\hbar\omega}{2}\left(\hat{\mathbf{a}}\hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}^\dagger\hat{\mathbf{a}}\right)$$

$$\left[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger\right] = \left(\hat{\mathbf{a}}\hat{\mathbf{a}}^\dagger - \hat{\mathbf{a}}^\dagger\hat{\mathbf{a}}\right) = \hat{\mathbf{1}} \rightarrow$$

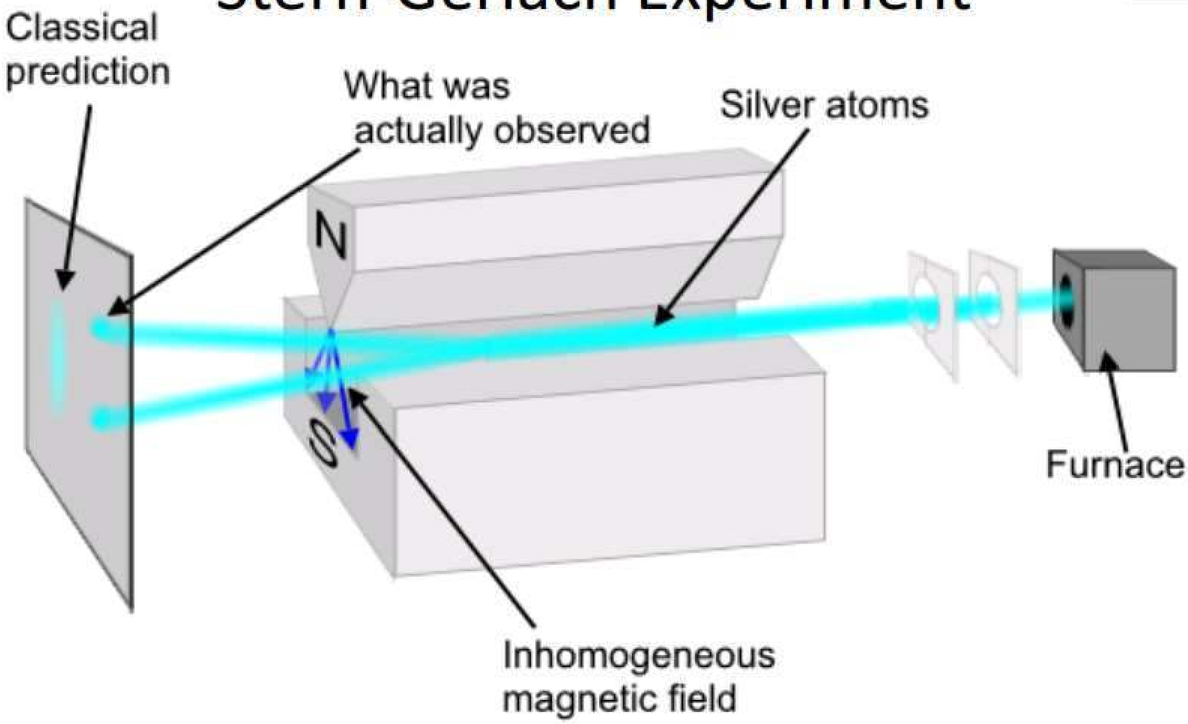
$$\hat{\mathbf{H}} = \frac{\hbar\omega}{2}\left(\hat{\mathbf{1}} + 2\hat{\mathbf{a}}^\dagger\hat{\mathbf{a}}\right) = \hbar\omega\left(\frac{1}{2} + \hat{\mathbf{a}}^\dagger\hat{\mathbf{a}}\right)$$

The energy levels of the harmonic oscillator are determined by the eigenvalues of the operator

$\hat{\mathbf{N}} = \hat{\mathbf{a}}^\dagger\hat{\mathbf{a}}$ called “number operator”.

stern

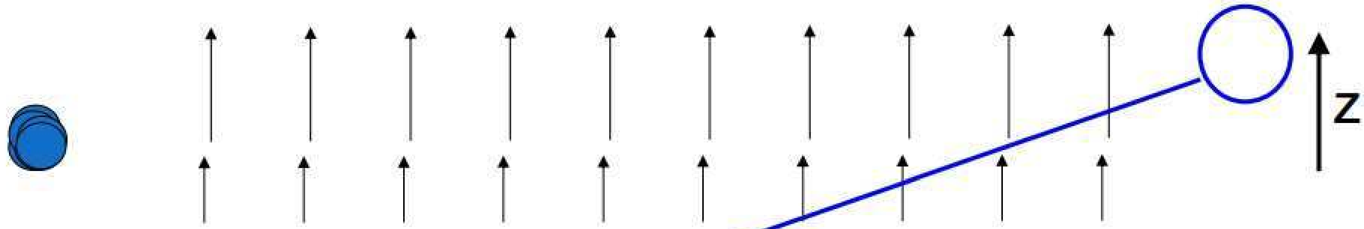
Stern-Gerlach Experiment



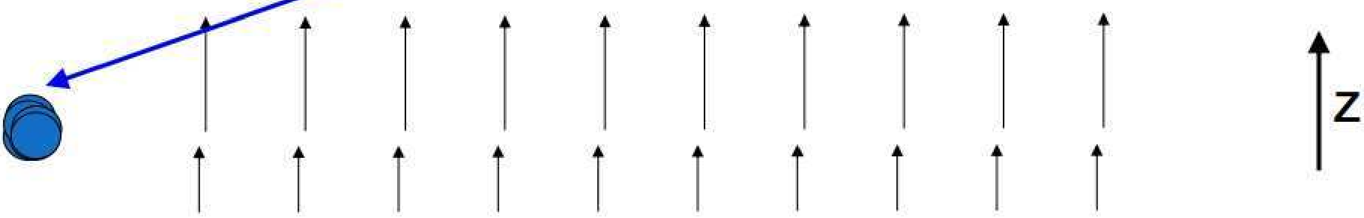
Ag: $\underbrace{1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 4s^2\ 3d^{10}\ 4p^6\ 4d^{10}}\ 5s^1$

Number of electrons $46 \quad + \quad 1 \quad = 47$

Put atoms in inhomogeneous magnetic field pointing in z direction – split in two groups – spin up and spin down

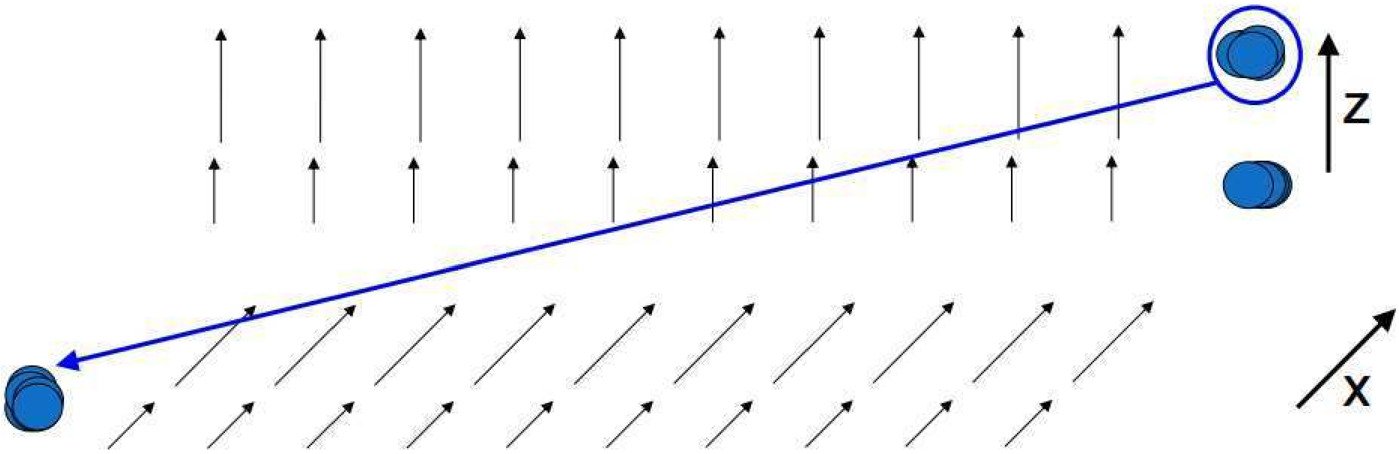


What if I take just atoms that went up, and send them through another, identical magnetic field – What happens?



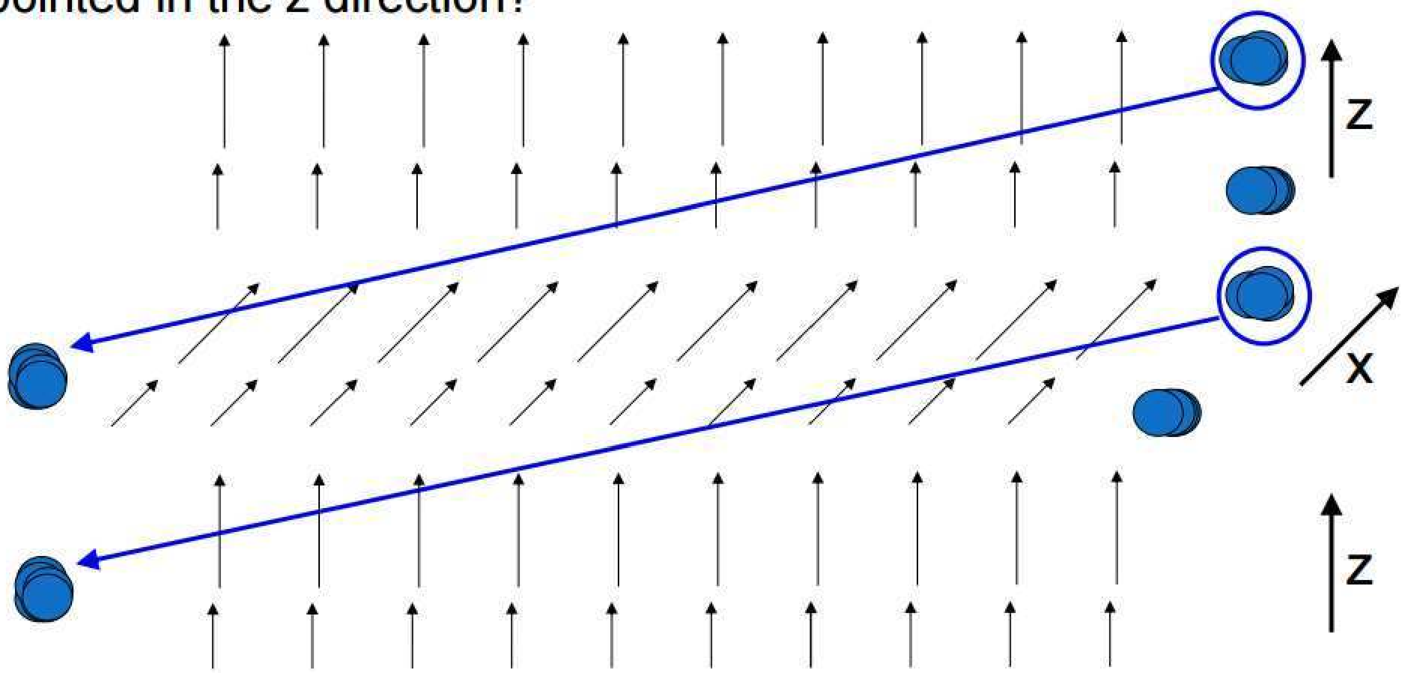
All go up (+z)

Second Experiment: What if I take just atoms that went up, and send them through a magnetic field pointed in the x direction – perpendicular to first field (pointing into the screen)?



Half go into the screen (+x), half go out of the screen (–x)

Third Experiment: Take just the atoms that went in +x direction in second experiment, and send them through a third magnetic field, pointed in the z direction?



Half go up (+z), half go down (–z).