3. The State-space of Quantum Mechanics: The Hilbert space

a. State space represented by ket and bra vectors of the Hilbert space. Observables represented by linear operators of the Hilbert space.

The Hilbert Space of Quantum Mechanics

Information we can know about a state

In quantum physics a ***physical state*** is represented by a ***state vector i***n a complex vector space, called ***Hilbert space***.

We call the state vector a ***„ket"***, and denote it by $|\alpha\rangle$

(This state ket is postulated to contain complete information about the physical state. Everything we are allowed to ask about the state is contained in the ket.)

Two kets can be added, and can be multiplied by a complex number, and the result is also a ket

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle \qquad \lambda|\alpha\rangle = |\alpha\rangle\lambda$$

Null ket $\lambda|\alpha\rangle$ if $\lambda = 0$

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$|\alpha\rangle$ and $\lambda|\alpha\rangle$, with $\lambda \neq 0$, represent the same physical state.

(Only the „direction" in vector space is of significance. We are dealing with rays rather than vectors .)

The state space of quantum physics is a Hilbert space

$$|a\rangle, |b\rangle \in \mathcal{H} \quad |a\rangle + |b\rangle = |c\rangle \in \mathcal{H}; \quad \lambda|a\rangle \in \mathcal{H} \quad \lambda \text{ complex number}$$

The dimension of the vector space depends on the physical system (Spin : 2; Finite dim.: *n*; Bounded: countable infinite; Free: continuously infinite)

Observable is represented by a linear operator

The operator acts on the ket from left, and maps a ket on a ket.

$$\hat{\mathbf{A}} \cdot (|a\rangle) = \hat{\mathbf{A}}|a\rangle \in \mathcal{H}$$

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b. Eigen-values and eigen-kets of an operator. Orthogonal vectors (kets). Set of orthonormal vectors.

In general $\hat{\mathbf{A}}|a\rangle \neq c \cdot |a\rangle$, but there are such kets

$$\hat{\mathbf{A}}\left|a^{(n)}\right\rangle = a^{(n)} \cdot \left|a^{(n)}\right\rangle \qquad \hat{\mathbf{A}}|n\rangle = a_{(n)} \cdot |n\rangle$$

$\left|a^{(1)}\right\rangle, \left|a^{(2)}\right\rangle, \ldots, \left|a^{(n)}\right\rangle, \ldots$ "eigen-kets" $\quad |1\rangle, |2\rangle, \ldots, |n\rangle, \ldots$

$a^{(1)}, a^{(2)}, \ldots, a^{(n)}, \ldots$ eigenvalues of operaot $\hat{\mathbf{A}}$

In an N-dimensional vector space
every ket can be expressed as

$$|a\rangle = \sum_n c_n \left|a^{(n)}\right\rangle \qquad |a\rangle = \sum_n c_n |n\rangle$$

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<u>c. The scalar product of a bra and a ket in Hilbert space. The outer product as operator.</u>

The Hilbert space is a linear vector space over the complex numbers, in which the scalar product of the elements exist.

Definition of the scalar product of a bra and a ket

$\langle b|a\rangle$ = number (in general complex), for which $\langle b|a\rangle = \langle a|b\rangle^*$

Scalar product is a „**bra**c**ket**"

$\langle a|a\rangle$ = real number; $\langle a|a\rangle \geq 0$; If $\langle a|a\rangle = 0 \rightarrow |a\rangle$ null ket

To kets are orthogonal

$$|a\rangle \perp |b\rangle \quad \text{if} \quad \langle a|b\rangle = 0$$

Normalized ket:

$$|\bar{a}\rangle = \frac{1}{\sqrt{\langle a|a\rangle}}|a\rangle \quad \rightarrow \langle \bar{a}|\bar{a}\rangle = 1$$

Bra	Ket	Bra-ket	Operator
$\langle b\vert$	$\vert a\rangle$	$\langle b\vert a\rangle$	$\hat{\mathbf{A}}$

Operators $\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}, \ldots \quad \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \ldots$

$$\hat{\mathbf{X}} = \hat{\mathbf{Y}} \quad \text{if} \quad \hat{\mathbf{X}}|a\rangle = \hat{\mathbf{Y}}|a\rangle \quad \text{for } \forall|a\rangle$$

null operator if $\hat{\mathbf{X}}|a\rangle = 0$ for $\forall|a\rangle$

$$\hat{\mathbf{X}} + \hat{\mathbf{Y}} = \hat{\mathbf{Y}} + \hat{\mathbf{X}}; \quad \hat{\mathbf{X}} + (\hat{\mathbf{Y}} + \hat{\mathbf{Z}}) = (\hat{\mathbf{X}} + \hat{\mathbf{Y}}) + \hat{\mathbf{Z}}$$

Linear Operators

$$\hat{\mathbf{X}}(c_a|a\rangle + c_b|b\rangle) = c_a\hat{\mathbf{X}}|a\rangle + c_b\hat{\mathbf{X}}|b\rangle$$

d. Adjoint, self-adjoint and projection operators in Hilbert space.

An operator acts on a bra from right, and maps a bra on a bra. $\langle a|\hat{\mathbf{X}}$

$$\hat{\mathbf{X}}|a\rangle \text{ and } \langle a|\hat{\mathbf{X}}$$

$\hat{\mathbf{X}}^\dagger$ is the adjoint of $\hat{\mathbf{X}}$ if $\hat{\mathbf{X}}|a\rangle$ and $\langle a|\hat{\mathbf{X}}^\dagger$ are dual

If $\hat{\mathbf{X}} = \hat{\mathbf{X}}^\dagger$ then they are self-adjoint (Hermitian)

Products of Operators In general $\hat{\mathbf{X}}\hat{\mathbf{Y}} \neq \hat{\mathbf{Y}}\hat{\mathbf{X}}$;

$$(\hat{\mathbf{X}}\hat{\mathbf{Y}})^\dagger = \hat{\mathbf{Y}}^\dagger\hat{\mathbf{X}}^\dagger; \quad \hat{\mathbf{X}}(\hat{\mathbf{Y}}\hat{\mathbf{Z}}) = (\hat{\mathbf{X}}\hat{\mathbf{Y}})\hat{\mathbf{Z}} = \hat{\mathbf{X}}\hat{\mathbf{Y}}\hat{\mathbf{Z}}$$

$$\hat{\mathbf{X}}(\hat{\mathbf{Y}}|a\rangle) = (\hat{\mathbf{X}}\hat{\mathbf{Y}})|a\rangle = \hat{\mathbf{X}}\hat{\mathbf{Y}}|a\rangle; \quad \langle b|\hat{\mathbf{X}}\hat{\mathbf{Y}} = \langle b|\hat{\mathbf{X}}(\hat{\mathbf{Y}}) = (\langle b|\hat{\mathbf{X}})\hat{\mathbf{Y}}$$

‚Outer' product $|a\rangle\langle b|$

e. Matrix representation of vectors (ket, bra) and operators in Hilbert space.

Matrix representation of kets bras and operators

In quantum physics the mathematical representation of observables are linear self-adjoint operators

Lemma 1. Eigenvalues of self-adjoint operators are real numbers. The eigen-kets belonging to different eigenvalues are orthogonal.

$$\hat{\mathbf{A}}|n\rangle = a^{(n)}\cdot|n\rangle \qquad a^{(1)},a^{(2)},\dots,a^{(n)},\dots \text{eigenvalues of } \hat{\mathbf{A}}$$

$$|1\rangle,|2\rangle,\dots,|n\rangle,\dots \text{"eigen-kets"}$$

$$\langle i|j\rangle = \delta_{ij}$$

Lemma 2. Eigen-kets constitute a complete orthonormal basis.

$$|a\rangle = \sum_n c^{(n)}|n\rangle \qquad c^{(n)} = \langle n|a\rangle$$

$$|a\rangle = \sum_n |n\rangle\langle n|a\rangle \qquad \sum_n |n\rangle\langle n| = \mathbf{1}$$

$$\langle a|a\rangle = \left\langle a\left|\sum_n |n\rangle\langle n|\right|a\right\rangle = \sum_n |\langle n|a\rangle|^2$$

If $|a\rangle$ is normalized, then $\sum_n |c_n|^2 = \sum_n |\langle n|a\rangle|^2 = 1$

Projection operator $\hat{\Lambda}_n = |n\rangle\langle n|$

$$\hat{\Lambda}_{a^{(n)}}|a\rangle = |n\rangle\langle n|a\rangle = c_n|n\rangle \qquad \sum_n \hat{\Lambda}_n = \hat{1} \quad \text{(Completeness)}$$

If N is the dimension of the space of kets, the representation of operator $\mathbf{X}$

$$\hat{\mathbf{X}} = \hat{\mathbf{X}} \to \hat{\mathbf{X}} = \left(\sum_{n=1}^{N}|n\rangle\langle n|\right)\hat{\mathbf{X}}\left(\sum_{m=1}^{N}|m\rangle\langle m|\right) =$$

$$= \sum_{m=1}^{N}\sum_{n=1}^{N}|m\rangle\langle m|\mathbf{X}|n\rangle\langle n|$$

$\langle m|\mathbf{X}|n\rangle$ N^2 number; $\langle m|$ is a row vector;

$|n\rangle$ is a column vector