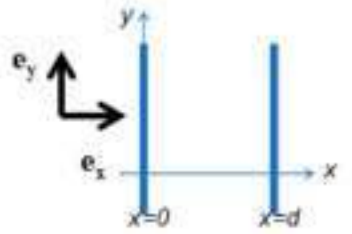
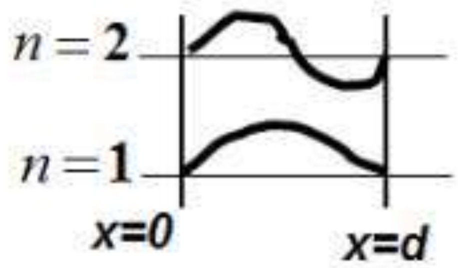
1. Wave-particle duality of the electromagnetic field (light).

a. Electromagnetic modes in an ideal cavity. (Plan-parallel mirrors. Rectangular shape ideal cavity)



The spatial eigen-value problem is $\Delta\mathbf{A}_n(\mathbf{r}) = -k_n^2\mathbf{A}_n(\mathbf{r})$, where $\mathbf{A}_n(\mathbf{r}) = A_n(x)\mathbf{e}_y$. The eigen-value problem is $\frac{d^2A_n(x)}{dx^2} = -k_n^2 A_n(x)$, and its general solution is $A_n(x) = c_1\sin k_n x + c_2\cos k_n x$, where c_1 and c_2 are arbitrary constants (to be determined by the boundary conditions). The electric field can be calculated from $\mathbf{E}_n(\mathbf{r},t) = -\frac{df_n(t)}{dt}\cdot\mathbf{A}_n(\mathbf{r})$. The direction of the electric field is $\mathbf{e}_y$, thus $A_n(0) = A_n(d) = 0$.

$$A_n(d) = 0 \rightarrow kd = n\cdot\pi \rightarrow k_n = \frac{n\cdot\pi}{d}, \quad n = 1,2,\ldots \qquad A_{k_n}(x) = c_1\sin\frac{n\pi}{d}x$$

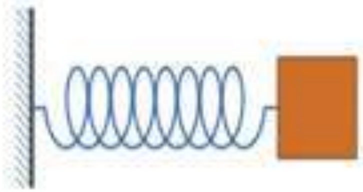


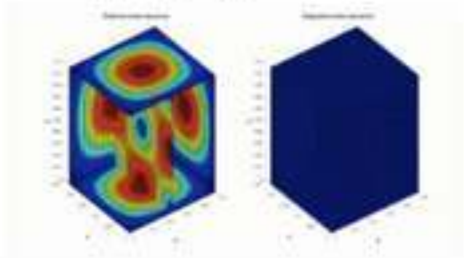
b. Analogy of a cavity mode with a quantum mechanical harmonic oscillator. Quantization of cavity modes. Photons.

The electromagnetic field is composed of resonant cavity modes with different frequencies.

We have shown that a cavity field of a given frequency mode is analog to a harmonic oscillator of the same frequency. To describe a field mode quantum mechanically, we simply describe the equivalent harmonic oscillator quantum mechanically.

The Lagrangian of a mechanical harmonic oscillator is $L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}Cx^2$, and its Hamiltonian is $H=\frac{1}{2}m\dot{x}^2+\frac{1}{2}Cx^2$. The resonant frequency is $\omega=\sqrt{\frac{C}{m}}$. A cavity mode Lagrangian is $L_k=\frac{1}{2}\varepsilon_0\dot{f}_k^2-\frac{k^2}{2\mu_0}f_k^2$, its Hamiltonian is $H_k=\frac{1}{2}\varepsilon_0\dot{f}_k^2+\frac{k^2}{2\mu_0}f_k^2$, and the resonant frequency of a rectangular cavity is $\omega_{m\ell p}=\pi\cdot c\sqrt{\frac{m^2}{a^2}+\frac{\ell^2}{b^2}+\frac{p^2}{L^2}};\quad m,\ell,p=1,2,3,\ldots$





The analogy of the harmonic oscillator and the a specific cavity mode suggests that to each cavity mode we can assign 1,2,3,...n,... energy eigen-values. In case of a mechanical harmonic oscillator of resonant frequency ω the quantized energies are

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right);n=1,2,\ldots$$

In case of a cavity mode of resonant frequency $\omega_{m\ell p}$ the quantized energies are

$$E_{(m\ell p)n}=\hbar\omega_{m\ell p}\left(n+\frac{1}{2}\right),\quad m,\ell,p=1,2,3,\ldots$$

The energy quantum $\hbar\omega_{m\ell p}$ is the energy of a photon with frequency $\omega_{m\ell p}$. In a cavity there are many resonant modes, and in each mode the energy is quantized, and each mode can have different number of photons.

Photons:

Photons are characterized by their frequency, free space wavelength, energy and wave- number and polarization. Particle nature (energy, momentum and spin) of the electromagnetic field becomes increasingly important as the frequency increases (Visible light, UV, X-rays, Gamma rays).

c. Vacuum fluctuations in a cavity. Casimir effect.

4. Vacuum Fluctuation

The energy of a cavity mode of frequency $\omega_{m\ell p}$ is quantized, and it can have only discrete values as $E_n^{m\ell p} = \hbar\,\omega_{m\ell p} \cdot \left(n + \frac{1}{2}\right); n = 1,2,3,...$. The energy of a photon is $\hbar\,\omega_{m\ell p}$. The total electromagnetic energy in a cavity is composed of the energy of an integer number photons, plus an energy, $\frac{1}{2}\hbar\,\omega_{m\ell p}$. This energy is called 'zero-point energy, or vacuum state energy. Vacuum state is a state which has no photons, but it nevertheless has an energy $\frac{1}{2}\hbar\,\omega_{m\ell p}$.

The expectation values of the vector potential as well as of the electric and magnetic fields are zero, however, the expectation values of the absolute squares of the field, i.e. the energy densities are not zero.

$$\langle \mathbf{A}(\mathbf{r},t)\rangle = 0, \quad \langle \mathbf{E}(\mathbf{r},t)\rangle = 0, \quad \langle \mathbf{B}(\mathbf{r},t)\rangle = 0; \quad \left\langle |\mathbf{E}|^2(\mathbf{r},t)\right\rangle > 0 \cdot$$

There is infinite number of resonant cavity modes in a cavity. Each of them has its zero- point energy. The fields fluctuate around the zero expectation values ('Vacuum fluctuations').

Casimir Effect:

There are many observable consequences of the vacuum field, which may be attributed at least in part to the vacuum field, such as spontaneous emission, Casimir effect, Lamb shift, van der Waals forces and the fundamental laser linewidth, as we shall see later.

Here we briefly mention the Casimir force between conducting plates. It is often cited as proof for the macroscopic reality of the vacuum electromagnetic field.

H. H. Casimir showed in 1948 that because of the vacuum fluctuation (zero point field) there is an attractive force between neutral parallel conducting plates (‚Casimir Effect'). According to a qualitative explanation of the plausibility, the Casimir force emerges between two conductive plates, because the energy density between the plates is smaller than outside of them, due to the fact that between the plates the electromagnetic boundary conditions prohibit certain modes which are present outside. The energy gradient is the force itself. If the two parallel plates are put inside a large cavity, with resonant frequencies,

$\omega_m = m\pi \cdot \frac{c}{a}$, but between the plates only modes with frequencies $\omega_n = n\pi \cdot \frac{c}{d}$ are allowed, and $a >> d$, then there will be much less modes between the plates that in the large cavity. The Casimir force can be calculated as $F = \frac{\pi \hbar c}{480} \frac{A}{d^4}$, where A is the surface of the parallel plates, d is the distance between the plates. If the distance is $d < 10\ \mathrm{nm}$, the pressure could be as big as $10^5\ \mathrm{Pa} = 1$ atm. The Casimir force between e.g. colloid particles is the largest force between neutral bodies in nature. Molecules which are neutral could attract or repel each other.