~~Infotod~~ előadás
Neurhálók

2015. 10. 13. / 1.

(+2) mi baj van még? nem ortogonálisak a mintáink általában

↑ eddig ilyesmiket feltételeztünk

$$y_l(k+1) = \mathrm{sgn}\left\{ \sum_{j=1}^{N} W_{lj}\, y_j(k) \right\} \qquad l = \text{index}$$

$$\bar{s}^{\beta} = \mathrm{sgn}\left\{ \bar{\bar{W}} \bar{s}^{\beta} \right\}$$

$$\bar{y}(0) = \bar{x}$$

$$W_{lj} = \frac{1}{N} \sum_{\alpha=1}^{M} s_i^{\alpha} s_j^{\alpha}$$

we have to choose ~~these~~ these randomly

$$P(s_i^{\alpha} = 1) = P(s_i^{\alpha} = -1) = 0.5$$

⇓

RNG

$$M \le \min\left\{ \frac{N}{2\pi} ; \frac{N}{2\log N} \right\}$$

capacity of the network

$$\bar{s}^{(1)} = (1\ -1\ -1\ \dots\ 1)$$

$$\bar{s}^{(2)} = (-1\ +1\ -1\ -1\ \dots\ 1)$$

$$\vdots$$

$$\bar{s}^{(M)} = \dots$$

Stored memory patterns / prototypes are not random (they have some meaning)

$$\bar{x} \rightarrow \boxed{\mathcal{T}\ (??)} \xrightarrow{\bar{z}} \boxed{\text{HN: } \bar{y}(k+1) = \mathrm{sgn}\{\bar{\bar{W}}\bar{y}(k)\}} \xrightarrow{\bar{q}^{\alpha}} \boxed{\text{LUT}} \rightarrow \bar{s}^{\beta}$$

$$W_{ij} = \frac{1}{N} \sum_{\alpha=1}^{M} q_i^{\alpha} q_j^{\alpha} \qquad P(q_i^{\alpha} = 1) = P(q_i^{\alpha} = -1) = 0.5$$

$$\bar{q}^{\alpha},\ \alpha = 1, \dots, M \leftarrow \text{RNG}$$

⇕ mapping

we map them

$$\bar{s}^{\alpha} \quad \alpha = 1, \dots, M$$

not random but deterministicly chosen which is dictated by the application

Topologically invariant mapping

$$d(\bar{x}, \bar{s}^{\beta}) < d(\bar{x}, \bar{s}^{\alpha}) \quad \forall \alpha,\ \alpha \ne \beta,\ \alpha = 1, \dots, M$$

$$\Rightarrow d(\bar{z}, \bar{q}^{\beta}) < d(\bar{z}, \bar{q}^{\alpha}) \quad \forall \alpha,\ \alpha \ne \beta,\ \alpha = 1, \dots, M$$

$$\le d(\bar{x}, \bar{s}^{\beta})$$

Implementation of the topologically invariant mapping

$$\mathcal{T}:\ z_l = \mathrm{sgn}\left\{ \sum_{j=1}^{N} M_{lj} x_j \right\} \quad l = 1, \dots, N \qquad M_{lj} := \frac{1}{N} \sum_{\alpha=1}^{M} q_l^{\alpha} \cdot s_j^{\alpha}$$

$$\bar{z} = \mathrm{sgn}\{ \bar{\bar{M}} \cdot \bar{x} \} \qquad \text{matrix-vector mult.} \qquad \bar{\bar{M}} = \frac{1}{N} \sum_{\alpha=1}^{M} \bar{q}^{\alpha} \cdot \bar{s}^{\alpha T}$$

normalized inner prod.

$$a(\bar{x}, \bar{s}^{\beta}) = \frac{1}{N} \sum^{N} x_i s_i^{\beta} = \frac{1}{N} \bar{x}^T \bar{s}^{\beta} \qquad a(\bar{z}, \bar{q}^{\beta}) = \frac{1}{N} \sum_{i=1}^{N} z_i q_i^{\beta}$$

We have to prove that: $d(\bar{x}, \bar{s}^{\beta}) \geq d(\bar{z}, \bar{q}^{\beta})$

$$a(\bar{x}, \bar{s}^{\beta}) \leq a(\bar{z}, \bar{q}^{\beta})$$

ha a skalárszorzat nagyobb, közelebb vannak

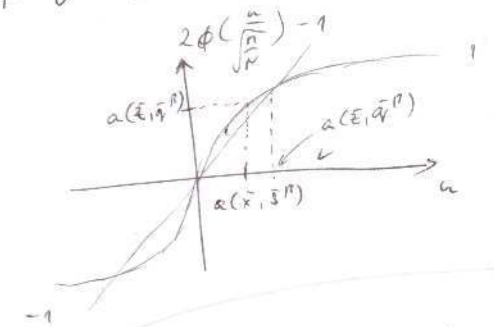
Let's assume that $q_i^{\beta} = 1 \quad \forall i = 1, \dots, N$

$$a(\bar{z}, \bar{q}^{\beta}) = \frac{1}{N}\sum_{i=1}^{N} z_i \, q_i^{\beta} = \frac{1}{N}\sum_{i=1}^{N} z_i \overset{n\to\infty}{\approx} E(z) = 1 \cdot P(z_i = 1) + (-1)\, P(z_i = -1)$$

$$\begin{aligned}
a(\bar{z}, \bar{q}^{\beta}) &= P(z_i = 1) - P(z_i = -1) = \\
&= P\left(\mathrm{sgn}\left\{ \sum_{j=1}^{N} M_{ij} x_j \right\} = 1 \right) - P\left(\mathrm{sgn}\left\{ \sum_{j=1}^{N} M_{ij} x_j \right\} = -1 \right) = \\
&= P\left(\mathrm{sgn}\left\{ \sum_{j=1}^{N} \frac{1}{N}\sum_{\alpha=1}^{M} q_i^{\alpha} s_j^{\alpha} x_j \right\} = 1 \right) - P\left(\dots = -1 \right) = \\
&= P\left(\mathrm{sgn}\left\{ \sum_{\alpha=1}^{M} q_i^{\alpha} \frac{1}{N}\sum_{j=1}^{N} s_j^{\alpha} x_j \right\} = 1 \right) - P\left(\dots = -1 \right) = \\
&= P\left(\mathrm{sgn}\left\{ q_i^{\beta} \frac{1}{N}\sum_{j=1}^{N} s_j^{\beta} x_j + \underbrace{\sum_{\substack{\alpha=1 \\ \alpha\neq\beta}}^{M} q_i^{\alpha} \frac{1}{N}\sum_{j=1}^{N} s_j^{\alpha} x_j}_{\xrightarrow{CLT} \sim \mathcal{N}\left(0, \sqrt{\frac{M(1)}{N}}\right)} \right\} = 1 \right) - P\left(\dots -1 \right) = \\
&= P\left(\mathrm{sgn}\left\{ a(\bar{x}, \bar{s}^{\beta}) + \nu_i \right\} = 1 \right) - P\left(\mathrm{sgn}\left\{ \bar{a}(\bar{x}, \bar{s}^{\beta}) + \nu_i \right\} = -1 \right) = \\
&= P\left(\bar{a}(\bar{x}, \bar{s}^{\beta}) + \nu_i > 0 \right) - P\left(\bar{a}(\bar{x}, \bar{s}^{\beta}) + \nu_i < 0 \right) \\
&= 2\,\phi\left(\frac{a(\bar{x}, \bar{s}^{\beta})}{\sqrt{\frac{M}{N}}} \right) - 1
\end{aligned}$$

$$a(\bar{z}, \bar{q}^{\beta}) = 2\,\phi\left(\frac{a(\bar{x}, \bar{s}^{\beta})}{\sqrt{\frac{M}{N}}} \right) - 1$$

topologically invariant if $\quad a(\bar{z}, \bar{q}^{\beta}) \geq a(\bar{x}, \bar{s}^{\beta})$



if $\quad M \leq \frac{N}{2\pi}$

Given $S = \{ \bar{s}^{\alpha}, \alpha = 1, \dots, M \} \quad M \leq \frac{N}{2\log N}$

$\Longrightarrow$ RNG $\quad Q = \{ \bar{q}^{\alpha}, \alpha = 1, \dots, M \}$ $\quad \bar{W} = \frac{1}{N}\sum_{\alpha=1}^{M} \bar{q}^{\alpha T} \bar{q}^{\alpha}$

$\bar{M}: \quad M_{\ell j} = \frac{1}{N}\sum_{\alpha=1}^{M} q_i^{\alpha} s_j^{\alpha}$

$\bar{x} \rightarrow \boxed{z_{\ell} = \mathrm{sgn}\left\{ \sum_{j=1}^{N} M_{\ell j} x_j \right\}} \xrightarrow{\bar{z} = \bar{y}(0)} \boxed{y_{\ell}(t+1) = \quad} \xrightarrow{\bar{q}^{\beta}} \boxed{\text{index}} \rightarrow \bar{s}^{\beta}$

$O(N^2)$

Hopfield net as a combinatorial optimizer

Intuitive description / idea: $\bar{y}(k+1) = \mathrm{sgn}\{\bar{W}\bar{y}(k) - \bar{b}\}$ $\xrightarrow[O(N^2)]{\text{stability}}$ $\alpha(\bar{y}) = \bar{y}^T\bar{W}\bar{y}^* - 2\bar{b}^T\bar{y}^*$

$\bar{y} \in \{1, -1\}^N$

optimization

~~(if I have a test)~~

the HN can be intepreted as a discrete quadratic optimizer ← real-time

↓

large family of problems

Quadratic programming:

optimization task of IT (routing tasks, discrete detection) → data representation → binary data repr. → QP ?? → $\min_{\bar{y}\in\{1,-1\}^N} \bar{y}^T\bar{W}\bar{y} - 2\bar{b}^T\bar{y}$ → Hopfield Net → $\bar{y}^*$

→ "De-representation" → solution of the original problem

~~[illegible]~~ we have to minimize instead of maximizing

How to minimize a Quadratic form by HN?

főátlót nullázzuk

Lemma. $\alpha(\bar{y}) = \bar{y}^T\bar{W}\bar{y} - 2\bar{b}^T\bar{y}$

$\bar{W}': W'_{ij} = \begin{cases} w_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$ $\quad \alpha'(\bar{y}) = \bar{y}\bar{W}'\bar{y} - 2\bar{b}^T\bar{y}$

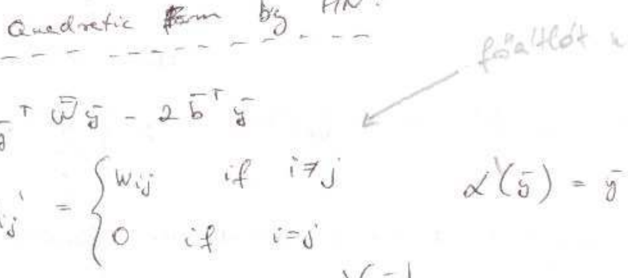
$\tilde{y}: \min_{\bar{y}\in\{-1,1\}^N} \alpha(\bar{y})$; $\bar{y}^*: \min_{\bar{y}\in\{-1,1\}^N} \alpha'(\bar{y})$

Proof: $\alpha(\bar{y}) = \bar{y}^T\bar{W}\bar{y} - 2\bar{b}^T\bar{y} = \sum_i\sum_j w_{ij}y_iy_j - 2\sum b_iy_i =$

$= \sum_i w_{ii}y_i^2 + \sum_i\sum_{j, i\neq j} w_{ij}y_iy_j - 2\sum_i b_iy_i = \sum_i w_{ii} + \sum_i\sum_j w'_{ij}y_i$

$- 2\sum_i b_iy_i = \text{const} + \alpha'(\bar{y}) \Rightarrow$ their minimums are in the same place

$y_e(k+1) = -\mathrm{sgn}\left\{\sum_{j=1}^N W'_{ej}\, y_j(k) - b_e\right\} \longrightarrow \min \alpha'(\bar{y})$



$TR \leq \frac{B - A\,k}{K}$

$$B = N \|\bar{\bar{W}}\| + 2\sqrt{N}\ \|\bar{b}\| \geq \alpha'(\bar{y})$$

$$A = -\bar{b}^T \bar{\bar{W}}^T \bar{b} \leq \alpha'(\bar{y})$$

$$\Delta\alpha'(\bar{y}(k)) = \alpha'(\bar{y}(k+1)) - \alpha'(\bar{y}(k)) = \sum_i \sum_j w_{ij}\ y_i(k+1)\, y_j(k+1) - 2\sum_i b_i\, y_i(k+1) =$$

$$- \sum_i \sum_j w'_{ij}\ y_i(k)\, y_j(k) + 2\sum_i b_i\, y_i(k)$$

$$\bar{y}(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_n(k) \end{pmatrix} \xrightarrow[l = \mathrm{mod}_N k]{} \begin{pmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_l(k+1) \\ \vdots \\ y_N(k) \end{pmatrix} = \bar{y}(k+1)$$

$$\Delta\alpha'(k) = \cancel{\Delta y_l^2(k)\, W'_{ll}}^{\,0} + 2\Delta y_l(k)\left\{\sum_{j=1}^{N} W_{lj}\ y_j(k) - b_l\right\}$$

$y_l(k)$	$y_l(k+1)$	$\sum_j W'_{lj}\ y_j(k) - b_l$	$2\Delta\ y_l(k)\left\{\sum_{j=1}^{N} W'_{lj}\ y_j(k) - b_l\right\} = \Delta\alpha(k)$	
-1	1	$\leq -K$	$-4K$	$2(1-(-1)) = 4$
1	-1	$\geq K$	$-4K$	$2(-1-(+1)) = -4$

always decreases

If $\alpha(\bar{y}) = \bar{y}^T \bar{\bar{W}} \bar{y} - 2\bar{b}^T \bar{y} \longrightarrow \bar{\bar{W}}' \longrightarrow y_l(k+1) = -\mathrm{sgn}\left\{\sum_{j=1}^{N} W'_{lj}\, y_j(k) - b_l\right\}$

Modified Hopfield Net can minimize quadratic forms over finite sets in P complexity.