Development of Complex Curricula for Molecular Bionics and Infobionics Programs within a consortial* framework**

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**Molekuláris bionika és Infobionika Szakok tananyagának komplex fejlesztése konzorciumi keretben

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Digital- and Neural Based Signal Processing & Kiloprocessor Arrays

Digitális- neurális-, és kiloprocesszoros architektúrákon alapuló jelfeldolgozás

Description digital signals and systems in transform (Z, DFT) domain

Digitális jelek és rendszerek leírása és analízise transzformált tartományokban

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LTI system as difference equation (review)

Every LTI system can be described in the time domain by its system equation which is a discrete time difference equation.

$$\sum_{k=0}^{N} a_k \cdot y(n-k) = \sum_{k=0}^{M} b_k \cdot x(n-k) \qquad \underbrace{\sum_{i=0}^{N} a_i S^i}_{D^N} y(n) = \underbrace{\sum_{j=0}^{M} b_j S^j}_{D^M} x(n)$$

without the loss of generality we usually assume $a_0 = 1$.

$$y(n) + a_1 \cdot y(n-1) + \cdots + a_N \cdot y(n-N) = b_0 \cdot x(n) + b_1 \cdot x(n-1) + \cdots + b_M\, x(n-M)$$

$$y(n) = -a_1 \cdot y(n-1) - \cdots - a_N \cdot y(n-N) + b_0 \cdot x(n) + b_1 \cdot x(n-1) + \cdots + b_M \cdot x(n-M)$$

This equation can be analyzed by well known mathematical analytical methods in the time domain.

Motivation of using transformed domains

- Certain transformations enable us to deal with the recurrence relation (difference equation) in the transformed domain as a simple algebraic equation.
- After solving the algebraic equations with simple mathematical tools we can transform back them into the to get the time domain response of the system

time domain — transformed domain

$\xrightarrow{\text{Transform}}$

difference equation — algebraic equation

Motivation of using transformed domains

- LTI systems can be fully characterized in the transformed domain by numbers (poles and zeroes).
- From these numbers we can simply tell important features of the system, e.g. stability, structural behavior, degree of the system, frequency characteristics.
- In continuous time we use integral transforms in discrete time we use power series, because these transforms have nice properties for derivation and convolution.
- LTI systems in the transformed domain will be represented by division of polynomials.

Geometric series (review)

- Note that at the time domain analysis we met with geometric sequences (e.g. homogenous solution) and when we apply the Laurent series the basic review of the geometric series will come in handy. We have the following closed forms for the series:

$$\sum_{k=0}^{n} c\cdot r^{k} = c\frac{1-r^{n+1}}{1-r}, \quad r \neq 1$$

$$\sum_{k=0}^{\infty} c\cdot r^{k} = \frac{c}{1-r}, \quad r < 1$$

$$\sum_{k=1}^{\infty} c\cdot r^{-k} = \frac{c}{r-1}, \quad r > 1$$

The z-transform

- The z-transform (bilateral) of a discrete time signal is defined as an infinite complex power series (Laurent series):

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}, \quad z \in \mathbb{C}$$

- For compactness we use the notation

$$\left.\begin{aligned} X(z) &= Z\{x(n)\} \\ x(n) &= Z^{-1}\{X(z)\} \end{aligned}\right\} x(n) \underset{Z^{-1}}{\overset{Z}{\rightleftarrows}} X(z)$$

for the z-transform and for the inverse z-transform.

The z-transform – example (entrant)

- Let's transform the following series into the z-domain:

$$x(n)=\begin{bmatrix}\dots & 0 & \underset{\uparrow}{5.2} & 3.1 & -15.2 & 2.4 & 0 & \dots\end{bmatrix}$$

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}, \quad z\in\mathbb{C}$$

$$X(z)=5.2z^{0}+3.1z^{-1}-15.2z^{-2}+2.4z^{-3}$$

$$=5.2+3.1\,{}^{1}\!/_{z}-15.2\,{}^{1}\!/_{z^{2}}+2.4\,{}^{1}\!/_{z^{3}}$$

Note that the series is convergent only for a region of z values. This example is convergent for $z\in\mathbb{C}, z\neq 0$

The z-transform – example (leaving)

- Let’s transform the following series into the z-domain:

$$x\left(n\right)=\left[\ldots \quad 0 \quad 5.2 \quad 3.1 \quad -15.2 \quad \underset{\uparrow}{2.4} \quad 0 \quad \ldots\right]$$

$$X\left(z\right)=\sum_{n=-\infty}^{\infty} x\left(n\right)\cdot z^{-n}, \quad z\in\mathbb{C}$$

$$X\left(z\right)=5.2z^{3}+3.1z^{2}-15.2z^{1}+2.4z^{0}$$

This example is convergent for $z\in\mathbb{C}, z\neq\infty$

- We have to define Region of Convergence.

The z-transform – RoC

- The region of convergence is defined as follows:

$$\mathrm{RoC}=\left\{ z\in\mathbb{C}: \left| \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n} \right| < \infty \right\}$$

all those numbers for which the series are convergent.

- Example 3: $x(n)=\begin{bmatrix} \dots & 0 & 5.2 & 3.1 & \underset{\uparrow}{-15.2} & 2.4 & 4.6 & 0 & \dots \end{bmatrix}$

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}, \quad z\in\mathbb{C}$$

$$X(z)=5.2z^{2}+3.1z^{1}-15.2z^{0}+2.4z^{-1}+4.6z^{-2}$$

$$\mathrm{RoC}=\left\{ z : z\in\mathbb{C}\setminus\{0,\infty\} \right\}$$

The z-transform – RoC

- We can decompose a signal to an entrant and to a leaving part:

$$x(n) = x_L(n) + x_E(n),$$

$$x_L(n) = \begin{cases} x(n) & n < 0 \\ 0 & n \geq 0 \end{cases}, \quad x_E(n) = \begin{cases} 0 & n < 0 \\ x(n) & n \geq 0 \end{cases}$$

The z-transform of a signal can be also decomposed to two parts:

$$X(z) = \sum_{n=-\infty}^{\infty} \left(x_L(n) + x_E(n)\right) \cdot z^{-n} = \sum_{n=-\infty}^{-1} x_L(n) \cdot z^{-n} + \sum_{n=0}^{\infty} x_E(n) z^{-n}$$

The z-transform – RoC

- The two parts have two different region of convergence

$$\left|X(z)\right| = \left|\sum_{n=-\infty}^{-1} x_L(n)\cdot z^{-n} + \sum_{n=0}^{\infty} x_E(n)\cdot z^{-n}\right| \le \sum_{n=-\infty}^{-1} \left|x_L(n)\right|\cdot\left|z\right|^{-n} + \sum_{n=0}^{\infty}\left|x_E(n)\right|\cdot\left|z\right|^{-n}$$

$$\left|X(z)\right| \le \sum_{n=-\infty}^{-1}\left|x_L(n)\right|\cdot r^{-n} + \sum_{n=0}^{\infty}\left|x_E(n)\right|\cdot r^{-n} = \underbrace{\sum_{n=1}^{\infty}\left|x_L(-n)\right| r^{n}}_{<\infty \text{ if } r<r_2} + \underbrace{\sum_{n=0}^{\infty}\left|x_E(n)\right| r^{-n}}_{<\infty \text{ if } r>r_1}$$

$\mathrm{RoC} = \left\{z : r_2 > \left|z\right| > r_1\right\}$ is an annulus for the general case

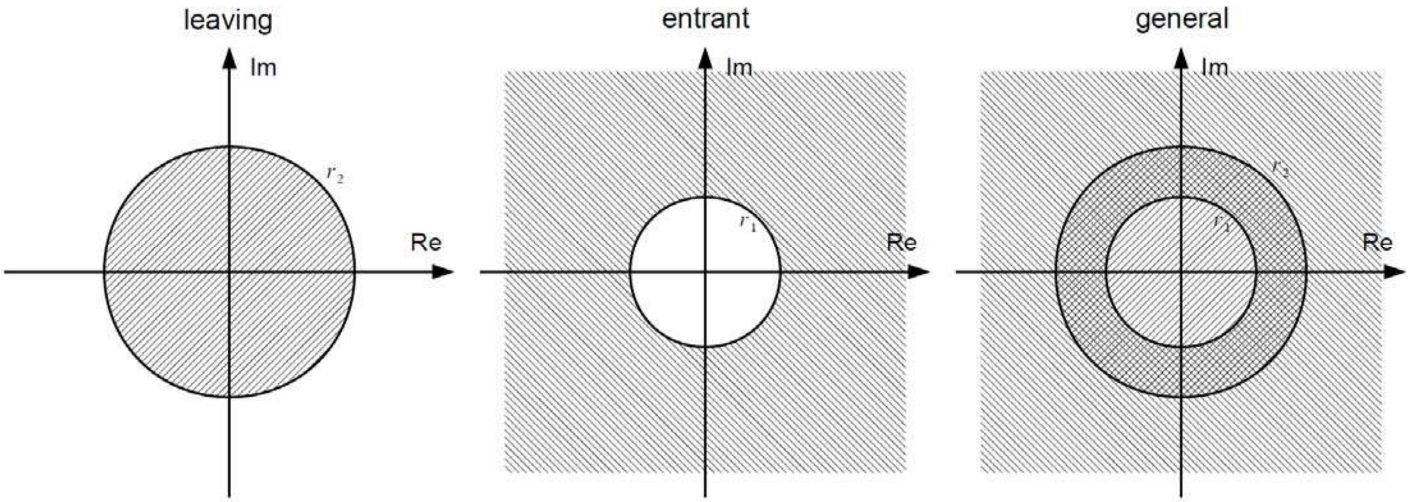
The z-transform – RoC

$\text{RoC} = \{z : |z| > r_1\}$ is a region outside of a circle for entrant signals

$\text{RoC} = \{z : r_2 > |z|\}$ is a region inside of a circle for leaving signals

for the general case the RoC is the intersection of the above two

$\text{RoC} = \{z : r_2 > |z| > r_1\}$ is an annulus for the general case



Properties of z-transform

property	time domain	Z-domain	RoC
linearity	$\alpha_1 x_1(n)+\alpha_2 x_2(n)$	$\alpha_1 X_1(z)+\alpha_2 X_2(z)$	$\mathrm{RoC}_1 \cap \mathrm{RoC}_2$
time shift	$x(n-k)$	$z^{-k}X(z)$	$\mathrm{RoC}\setminus\{0\}, k>0$ $\mathrm{RoC}\setminus\{\infty\}, k<0$
scaling in the z-domain	$a^n \cdot x(n)$	$X\left(a^{-1}\cdot z\right)$	$\lvert a\rvert r_2 > \lvert z\rvert > \lvert a\rvert r_1$
time reversal	$x(-n)$	$X\left(z^{-1}\right)$	$1/r_2 > \lvert z\rvert > 1/r_1$
Differentiation in z-domain	$n\cdot x(n)$	$-z\cdot\frac{d}{dz}X(z)$	RoC
accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$	
Convolution	$x_1(n)*x_2(n)$	$X_1(z)\cdot X_2(z)$	$\mathrm{RoC}_1 \cap \mathrm{RoC}_2$

Review of poles-zeroes (complex analysis)

Let's assume that $f(z)$ is analytic

zero: if $f(z_0)=0$ and we can write $f(z)=(z-z_0)^n\,\tilde{f}(z)$,

$\tilde{f}(z)$ analytic, $\tilde{f}(z_0)\neq 0, \exists n \geq 1$

then z_0 is a zero of order n

pole: if we can write $f(z)=\dfrac{1}{(z-z_0)^n}h(z)$,

$h(z)$ analytic, $h(z_0)\neq 0, \exists n \geq 1$

then z_0 is a pole of order n

Laurent series and inverse z-transform

Laurent series:

if $f(z)$ is analytic in an annulus $D=\{z: r_1<|z-z_0|<r_2, r_1<r_2\}$ then

$f(z)$ can be expanded into a power series $f(z)=\sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$, where

$c_k=\frac{1}{2\pi j}\oint_C \frac{X(z)}{(z-z_0)^{k+1}}dz$, where C is a closed path encircling z_0, and in D

inverse z-transform

$$x(n)=Z^{-1}\{X(z)\}:=\frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

C is a closed path in the complex plane encircling the origin, all the poles of $X(z)$ and it must lie eintirely within the RoC.

Inverse z-transform in practice

- In an LTI system we expect system exponential type responses (see time domain analysis homogenous solution) and excitation like answers (see time domain analysis particular part).
- So usually it is enough if we have a table of the most common functions z-transforms.
- And then we need to manipulate them into correct form and reverse them into the time domain from a table. Thus we don't need to evaluate the complex closed path integral.

Z-transform of elementary DT signals

- We will present the z-transform, the visualization and the pole-zero plot of some elementary functions:

time	z-domain	RoC
$\delta(n)$	1	$z \in \mathbb{C}$
$u(n)$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
$u(-n)$	$\frac{1}{1-z}$	$\lvert z\rvert < 1$

time	z-domain	RoC
$n \cdot u(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$\cos(n \cdot \omega_0)$	$\frac{z^2 - z\cos(\omega_0)}{z^2 - 2z\cos(\omega_0) + 1}$	$\lvert z\rvert > 1$
$\sin(n \cdot \omega_0)$	$\frac{z\sin(\omega_0)}{z^2 - 2z\cos(\omega_0) + 1}$	$\lvert z\rvert > 1$

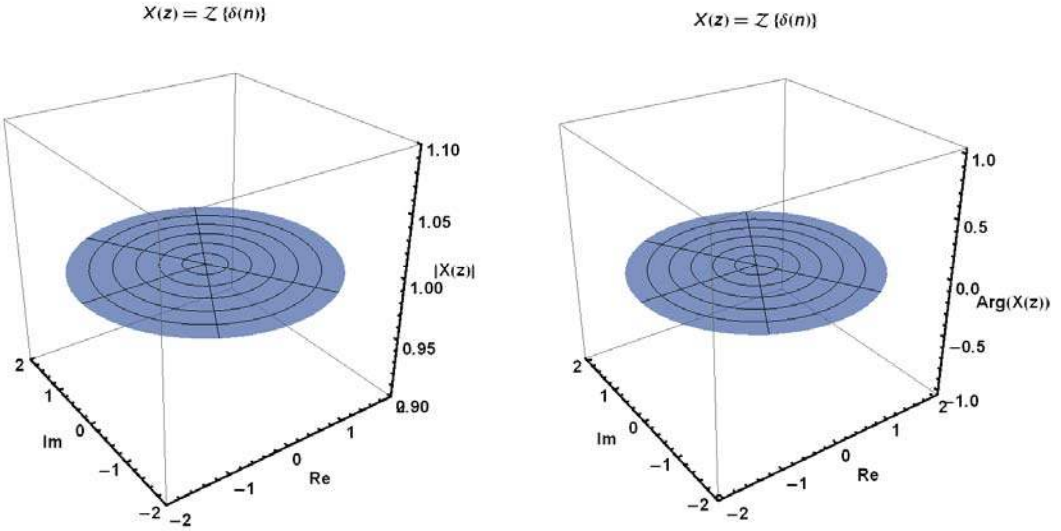
- Pole-zero plot of an LTI system

Z-transform of elementary DT signals

- Kronecker delta or unit sample

$$x(n)=\delta(n)=\begin{cases}1, & \text{if } n=0\\ 0, & \text{otherwise}\end{cases} \qquad Z\{\delta(n)\}=1, \quad \text{RoC}=\{z:z\in\mathbb{C}\}$$

$$X(z)=\sum_{n=-\infty}^{\infty}\delta(n)\cdot z^{-n}=z^{0}=1$$



Z-transform of elementary DT signals

- Unit step function

$$x(n)=u(n)=\begin{cases}1, & \text{if } n\geq 0\\ 0, & \text{otherwise}\end{cases} \qquad Z\{u(n)\}=\frac{1}{1-z^{-1}}=\frac{z}{z-1}, \quad \text{ROC}=\{z:|z|>1\}$$

$$X(z)=\sum_{n=-\infty}^{\infty}u(n)\cdot z^{-n}=\sum_{n=0}^{\infty}z^{-n}\underset{\substack{\text{geom.series}\\ \text{assumption}}}{=}\frac{1}{1-z^{-1}}$$

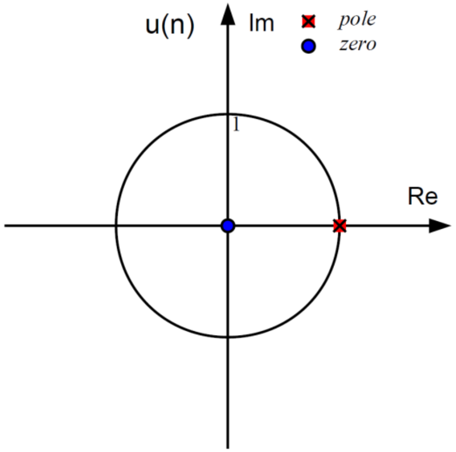
RoC=geometric series closed form only valid if $\left|z^{-1}\right|<1\Rightarrow \text{ROC}=\{z:|z|>1\}$

pole-zero plot:

poles and zeroes of a conplex function $X(z)$

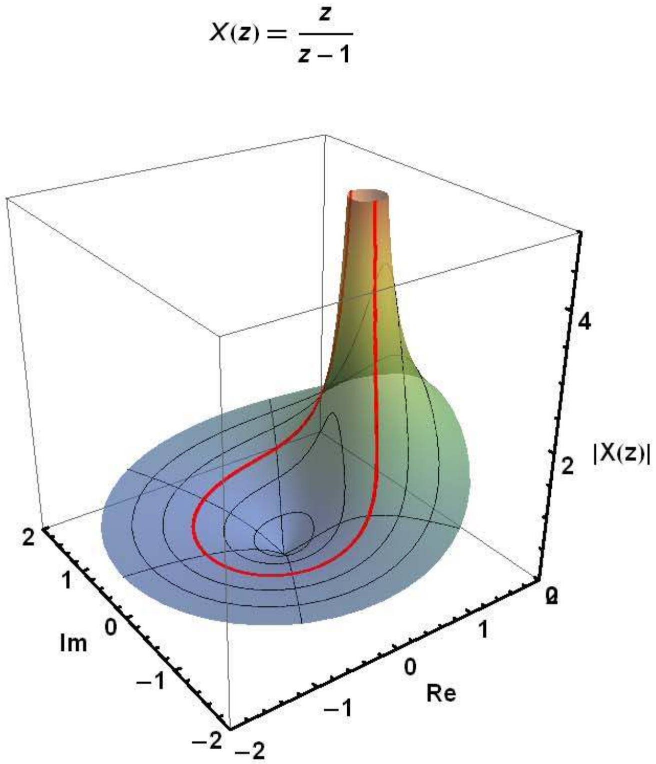
pole $\cong\{z:X(z)=\pm\infty\}$

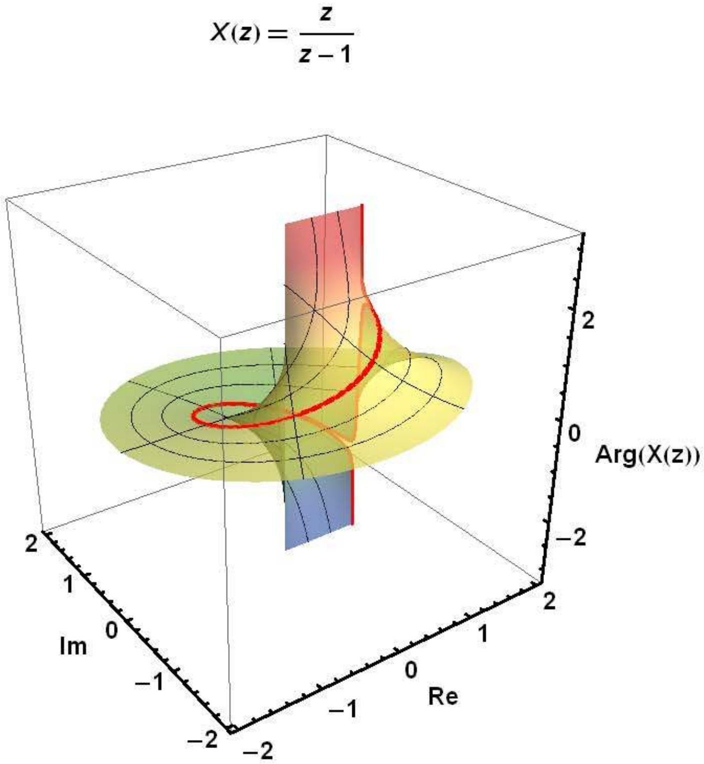
zero $\cong\{z:X(z)=0\}$



Z-transform of elementary DT signals

- Plot of function $X(z)=\frac{1}{1-z^{-1}}=\frac{z}{z-1}$, red line=$\{X(z):|z|=1\}$

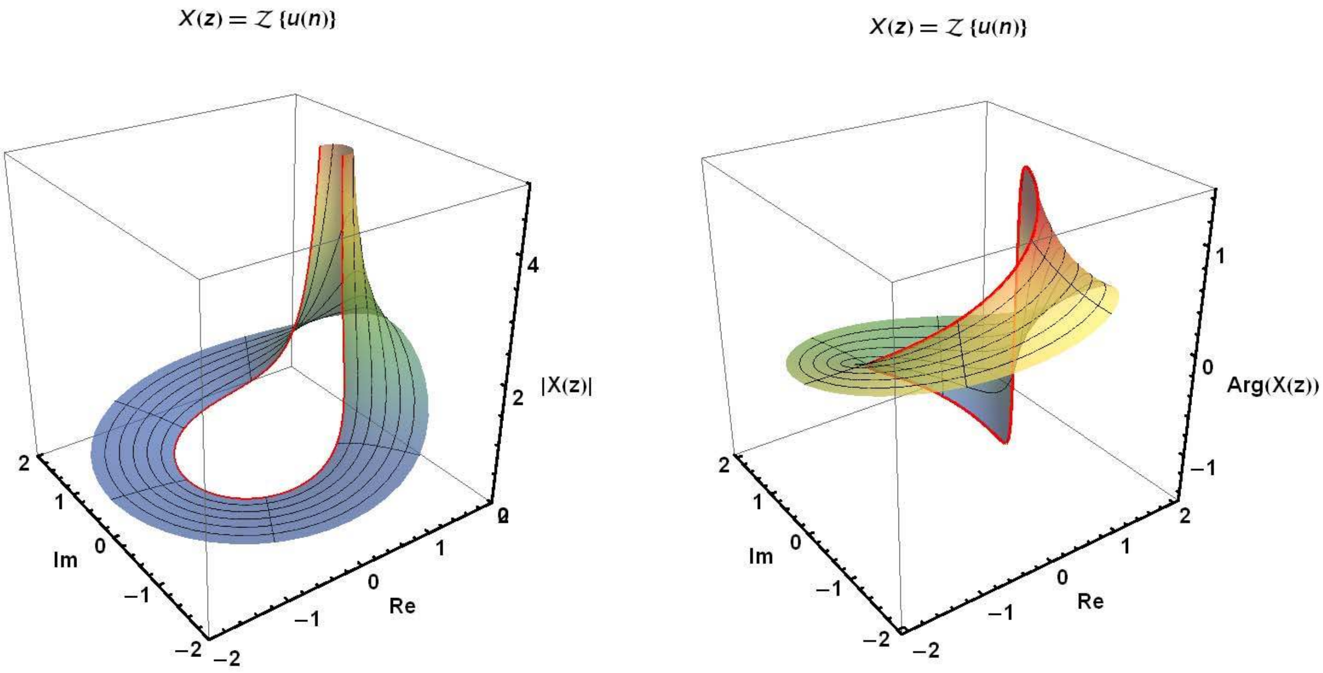




Z-transform of elementary DT signals

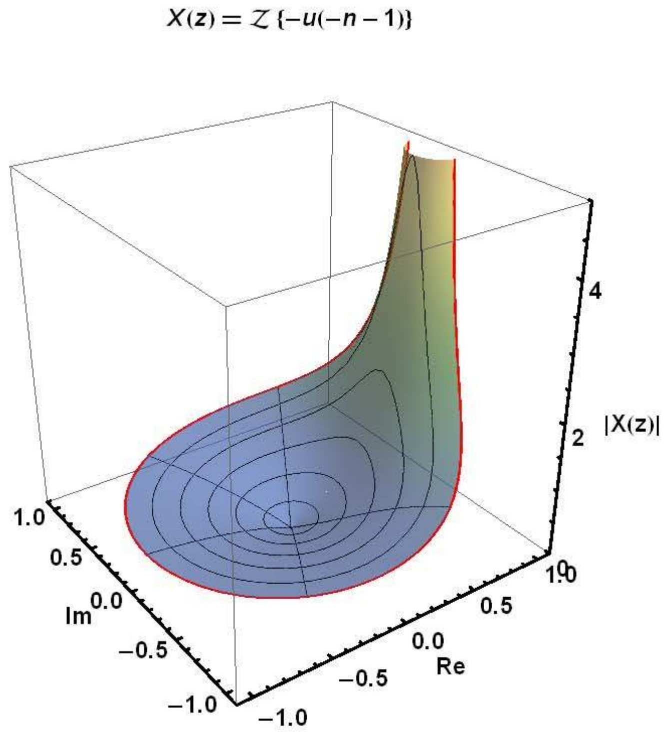
- Plot of function $x(n)=u(n) \quad Z\{u(n)\}=\frac{1}{1-z^{-1}}=\frac{z}{z-1}, \quad \text{RoC}=\{z:|z|>1\}$

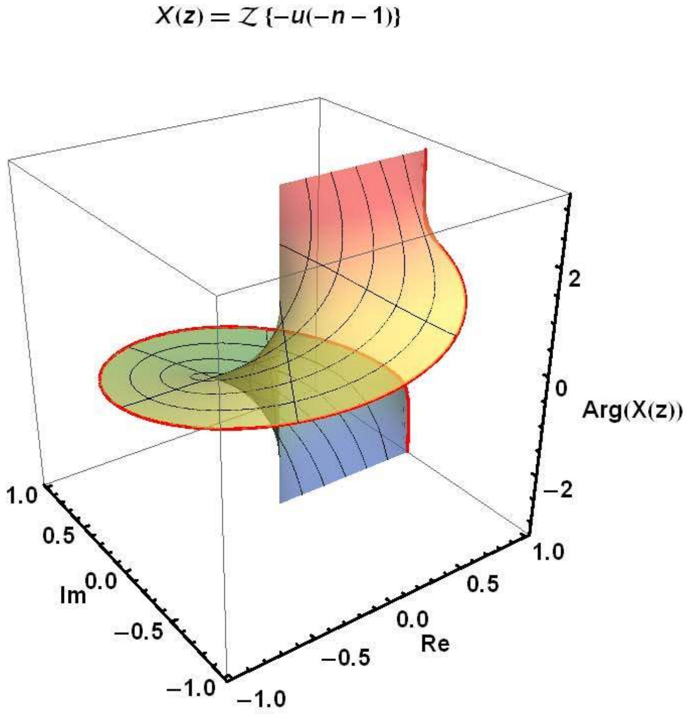
(note that outside the RoC the function is not determined)



Z-transform of elementary DT signals

- Plot of function $x(n) = -u(-n-1)$ $\quad Z\{-u(-n-1)\} = \dfrac{z}{z-1}, \quad \text{RoC}=\{z : |z| < 1\}$

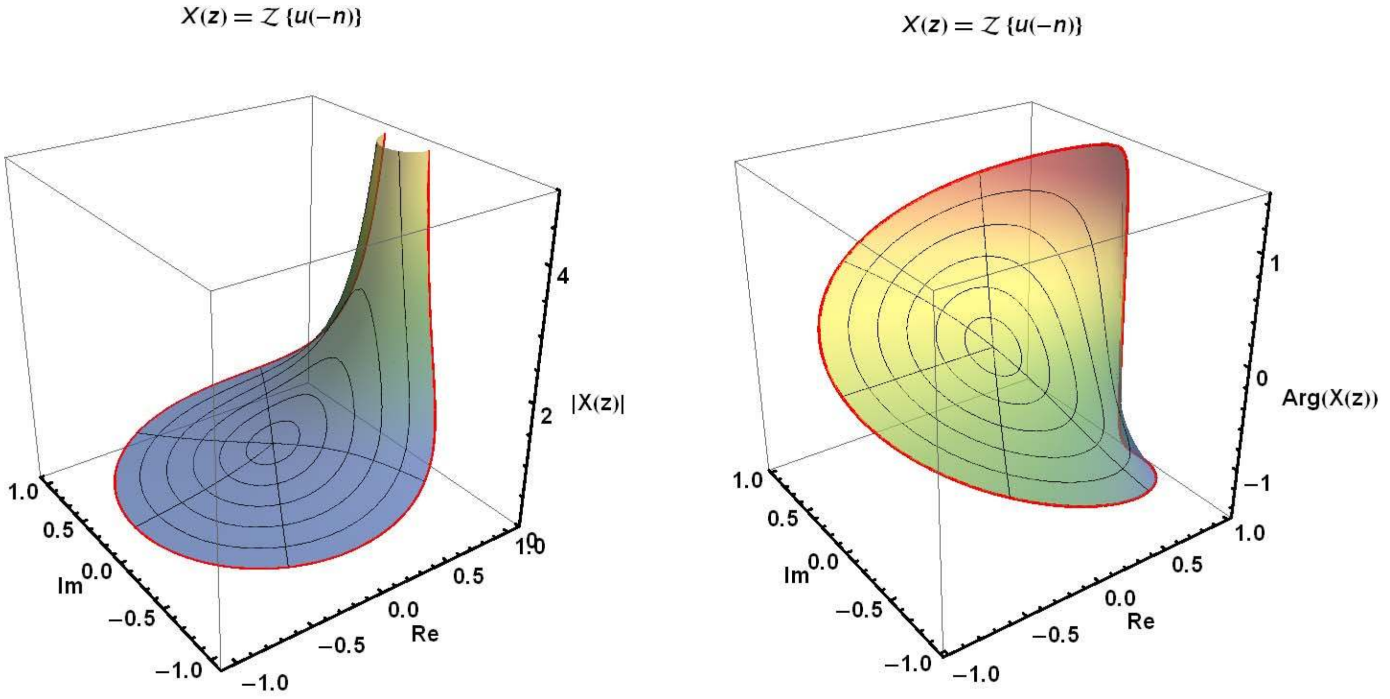




Z-transform of elementary DT signals

- Plot of function $x(n)=u(-n)\quad Z\{u(-n)\}=\dfrac{1}{1-z},\quad \mathrm{RoC}=\{z:|z|<1\}$

using the time reversal property



Z-transform of elementary DT signals

- Unit ramp function

$$x(n)=u_r(n)=\begin{cases} n, & \text{if } n\geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad Z\{u_r(n)\}=\frac{z^{-1}}{\left(1-z^{-1}\right)^2}=\frac{z}{(z-1)^2}, \quad \text{RoC=}\{z:|z|>1\}$$

we use the property of differentiation in z-domain: $-z\frac{d}{dz}Z\{u(n)\}=Z\{n\cdot x(n)\}$

$$u_r(n)=n\cdot u(n)$$

$$X(z)=Z\{n\cdot u(n)\}=-z\frac{d}{dz}Z\{u(n)\}=-z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)=-z\left(-\frac{z^{-2}}{\left(1-z^{-1}\right)^2}\right)$$

$X(z)=\frac{z^{-1}}{\left(1-z^{-1}\right)^2}$ RoC=$\{z:|z|>1\}$ same as the unit step function.

Z-transform of elementary DT signals

- Pole-zero plot of the unit ramp function in the z-domain

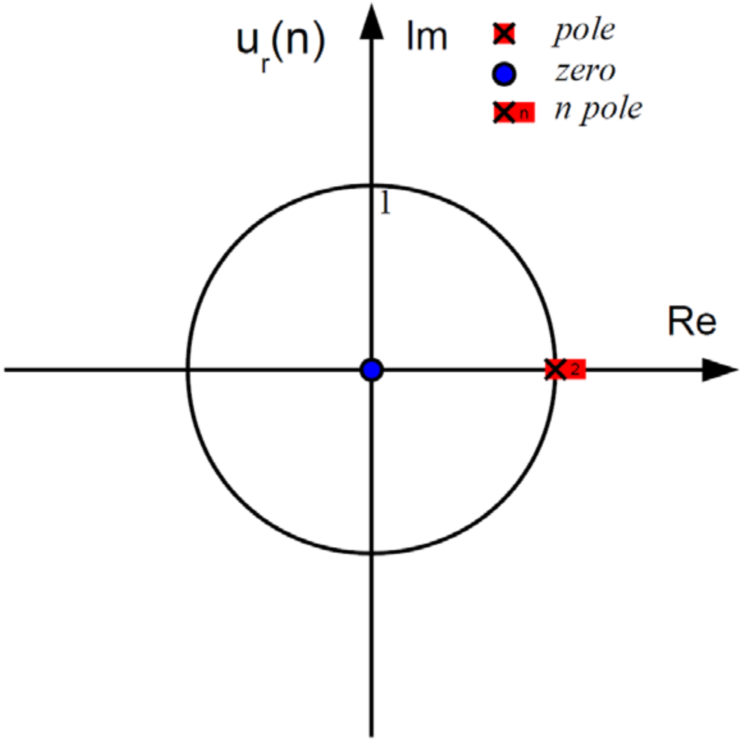
poles and zeroes of the function:

$$X\left(z\right)=\frac{z}{\left(z-1\right)^{2}}$$

$$\left.\begin{array}{l} p_1=1 \\ p_2=1 \end{array}\right\}\text{multiplicative roots of } \left(z-1\right)^{2}$$

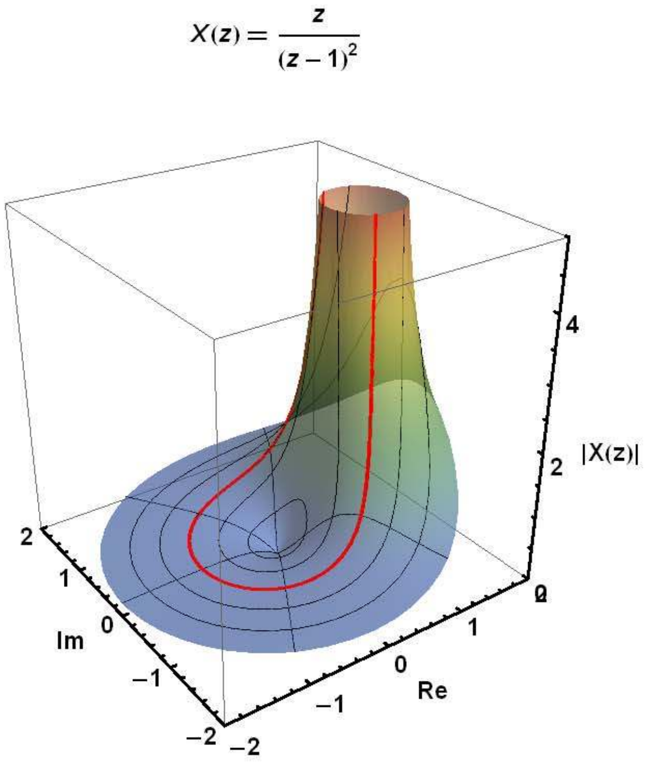
note: **resonance** from time domain analysis

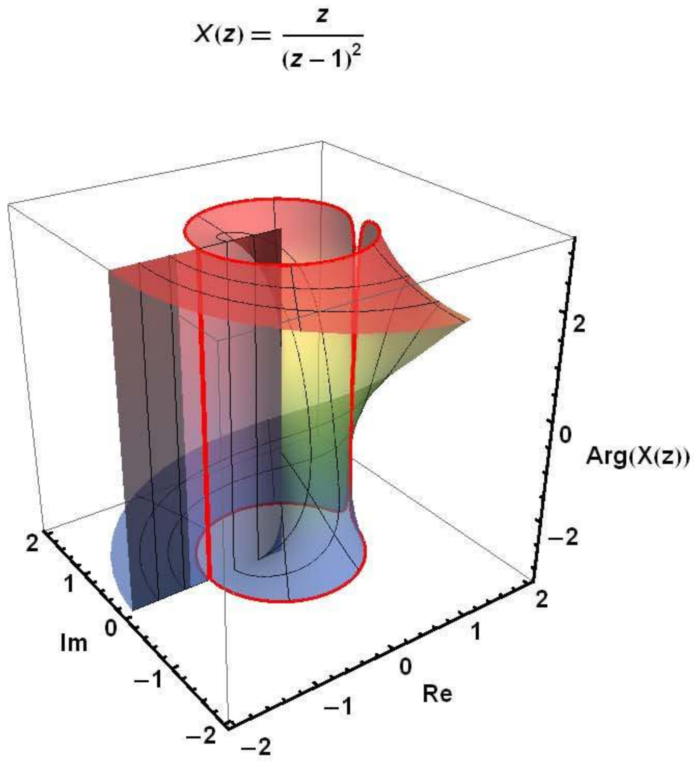
$z_1=0$ root of z



Z-transform of elementary DT signals

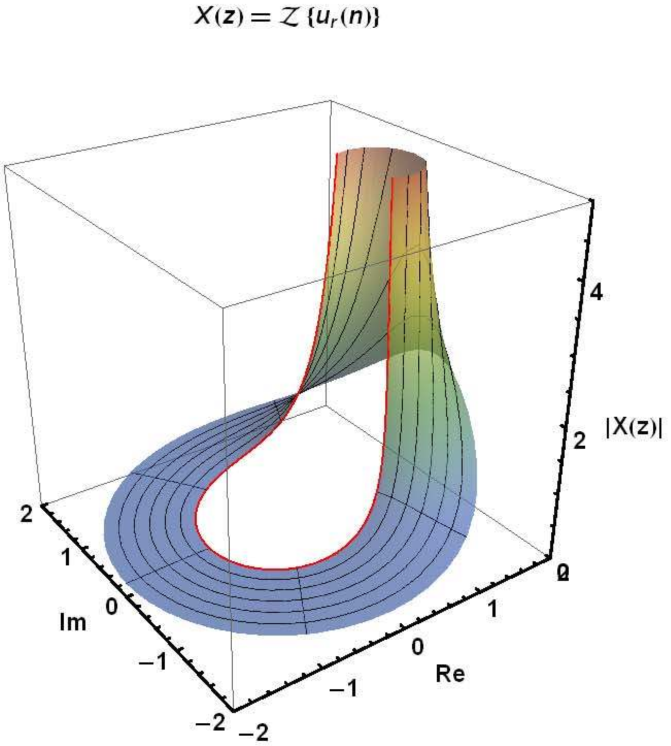
- Plot of function $X(z)=\frac{z}{(z-1)^2}$, red line$=\{X(z):|z|=1\}$

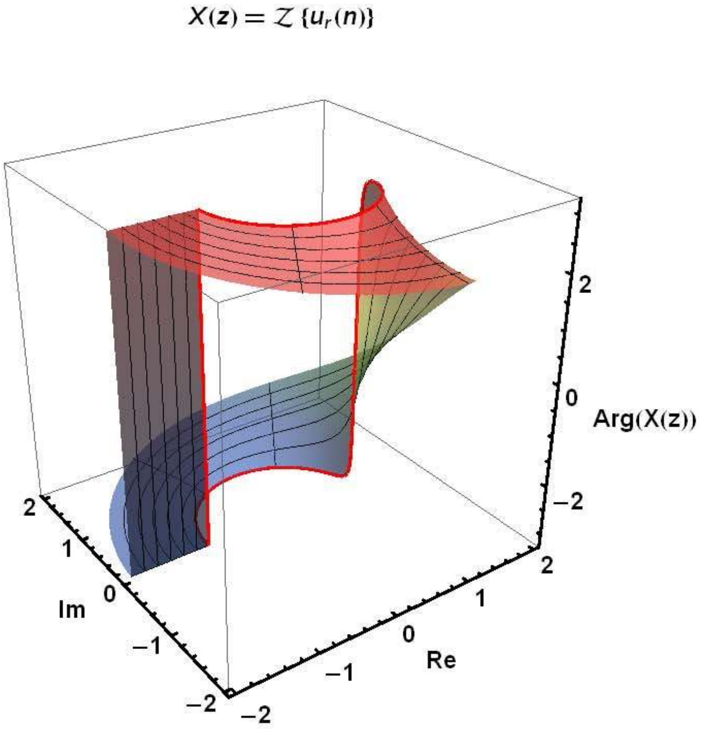




Z-transform of elementary DT signals

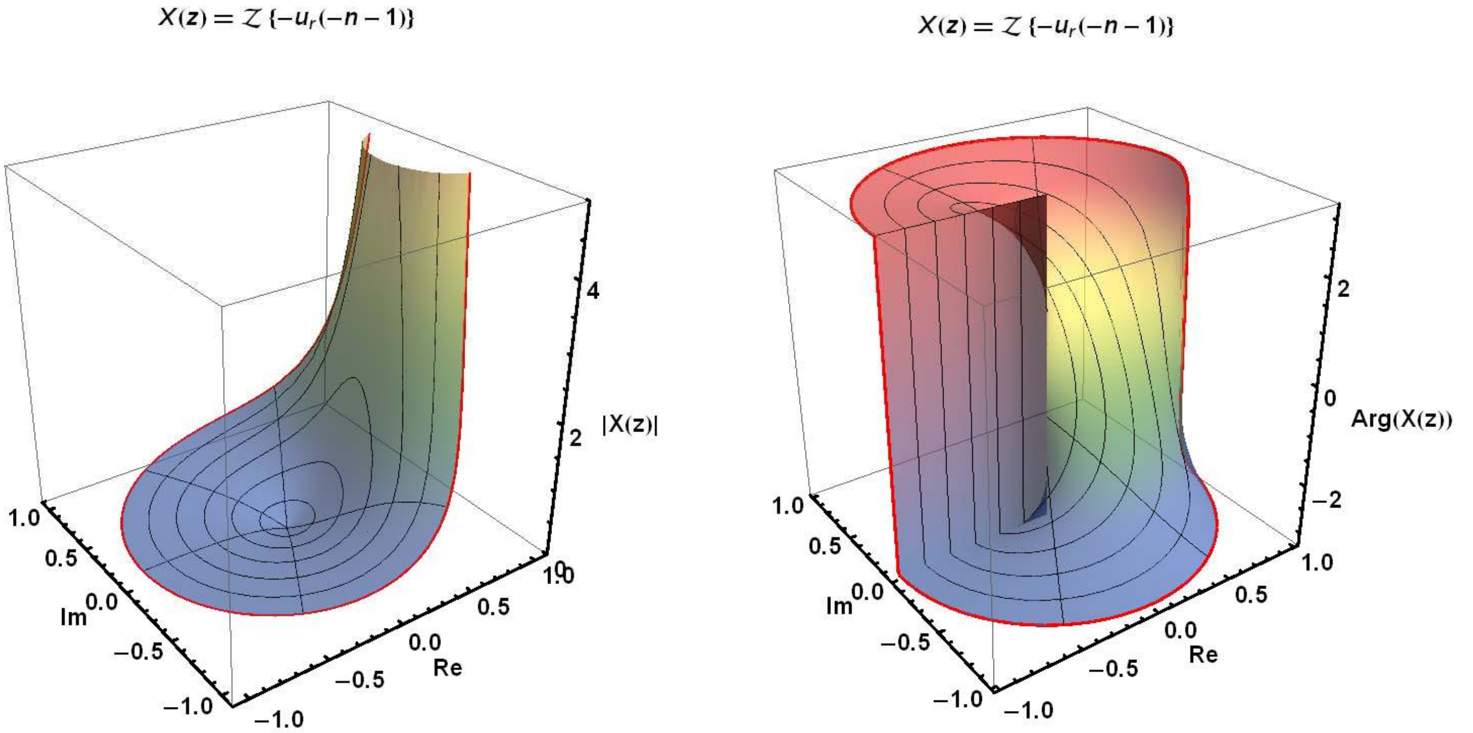
- Plot of function $x(n)=u_r(n) \quad Z\{u_r(n)\}=\dfrac{z}{(z-1)^2}, \quad \text{RoC}=\{z:|z|>1\}$





Z-transform of elementary DT signals

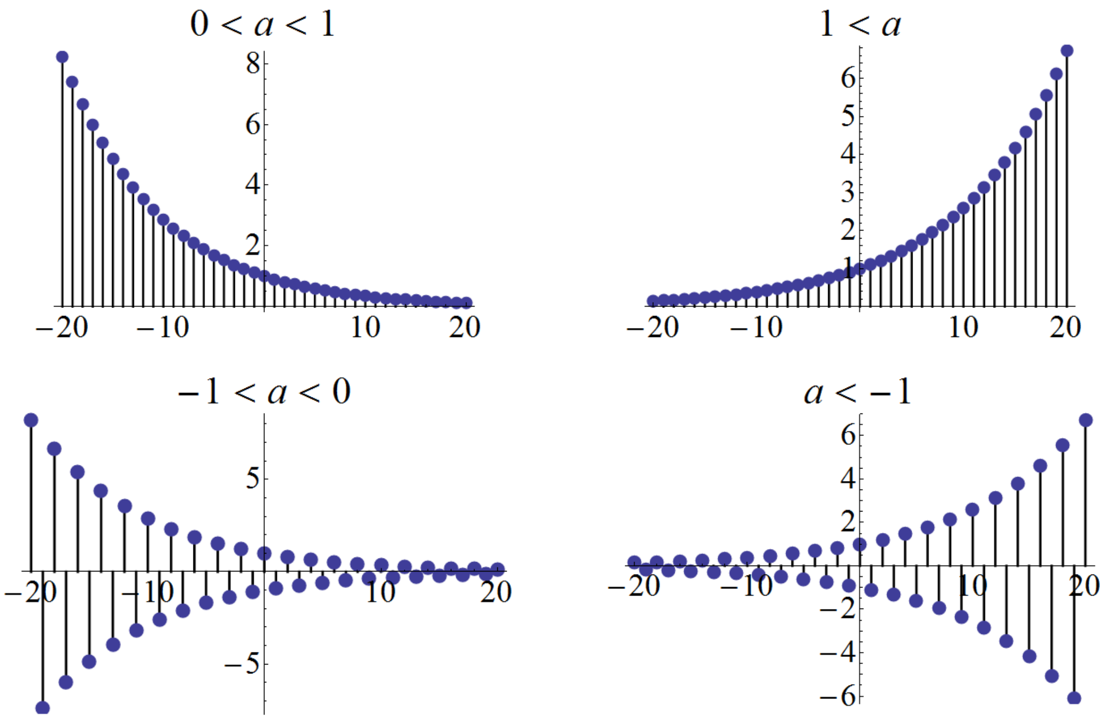
- Plot of function $x(n) = -u_r(-n-1) \quad Z\{x(n)\} = \frac{z}{(z-1)^2}, \quad \text{RoC=}\{z : |z| < 1\}$



Z-transform of elementary DT signals

- Exponential function

$$x(n)=a^n,\quad \text{if } a\in\mathbb{R},\rightarrow x(n)\in\mathbb{R},\quad \text{if } a\in\mathbb{C},\rightarrow x(n)=\left(r\cdot e^{j\varphi}\right)^n=r^n\cdot e^{j\varphi n}$$



Z-transform of elementary DT signals

- z-transform of the exponential function

$$x(n)=a^n u(n),\quad Z\{x(n)\}=\frac{1}{1-a\cdot z^{-1}}=\frac{z}{z-a},\quad \text{RoC=}\{z:|z|>|a|\}$$

using the scaling in the z domain property $a^n \cdot x(n) \rightleftharpoons X\left(a^{-1}z\right)$

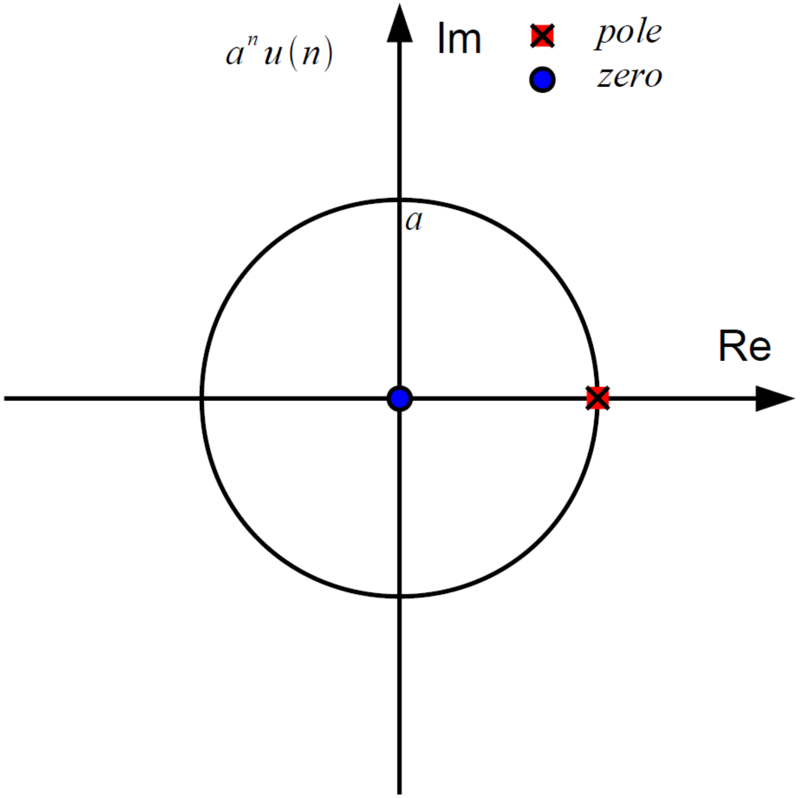
$$Z\{a^n x(n)\}=\sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}=\sum_{n=-\infty}^{\infty} x(n)\left(a^{-1}z\right)^{-n}=X\left(a^{-1}z\right)$$

$$Z\{u(n)\}=\frac{1}{1-z^{-1}},\quad \text{RoC=}\{z:|z|>1\}$$

$$Z\{a^n u(n)\}=\frac{1}{1-\left(a^{-1}z\right)^{-1}}=\frac{z}{z-a},\quad \text{RoC=}\{z:|z|>|a|\}$$

Z-transform of elementary DT signals

- Pole-zero plot of the exponential function in the z-domain



poles and zeroes of the function:

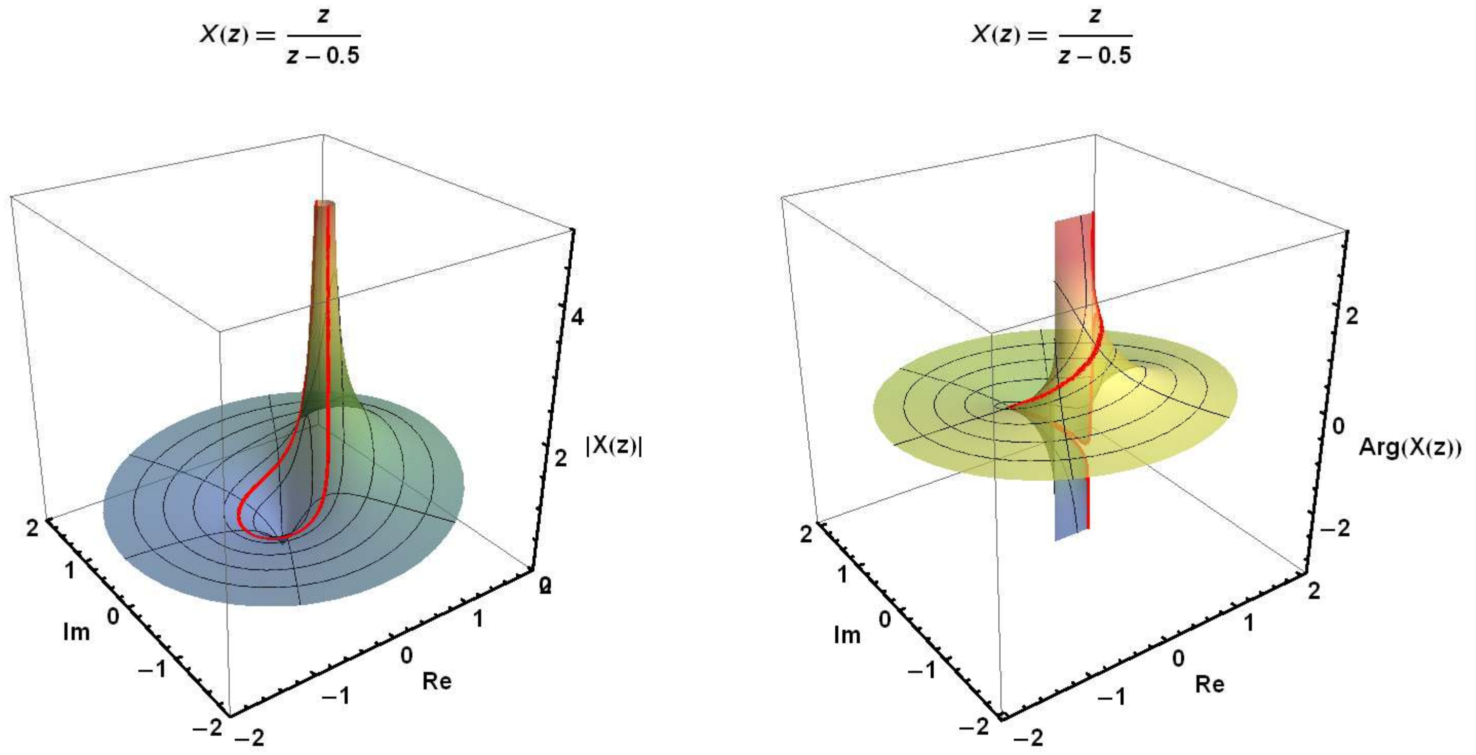
$$X(z)=\frac{z}{z-a}$$

$p_1 = a$ root of $z-a$

$z_1 = 0$ root of z

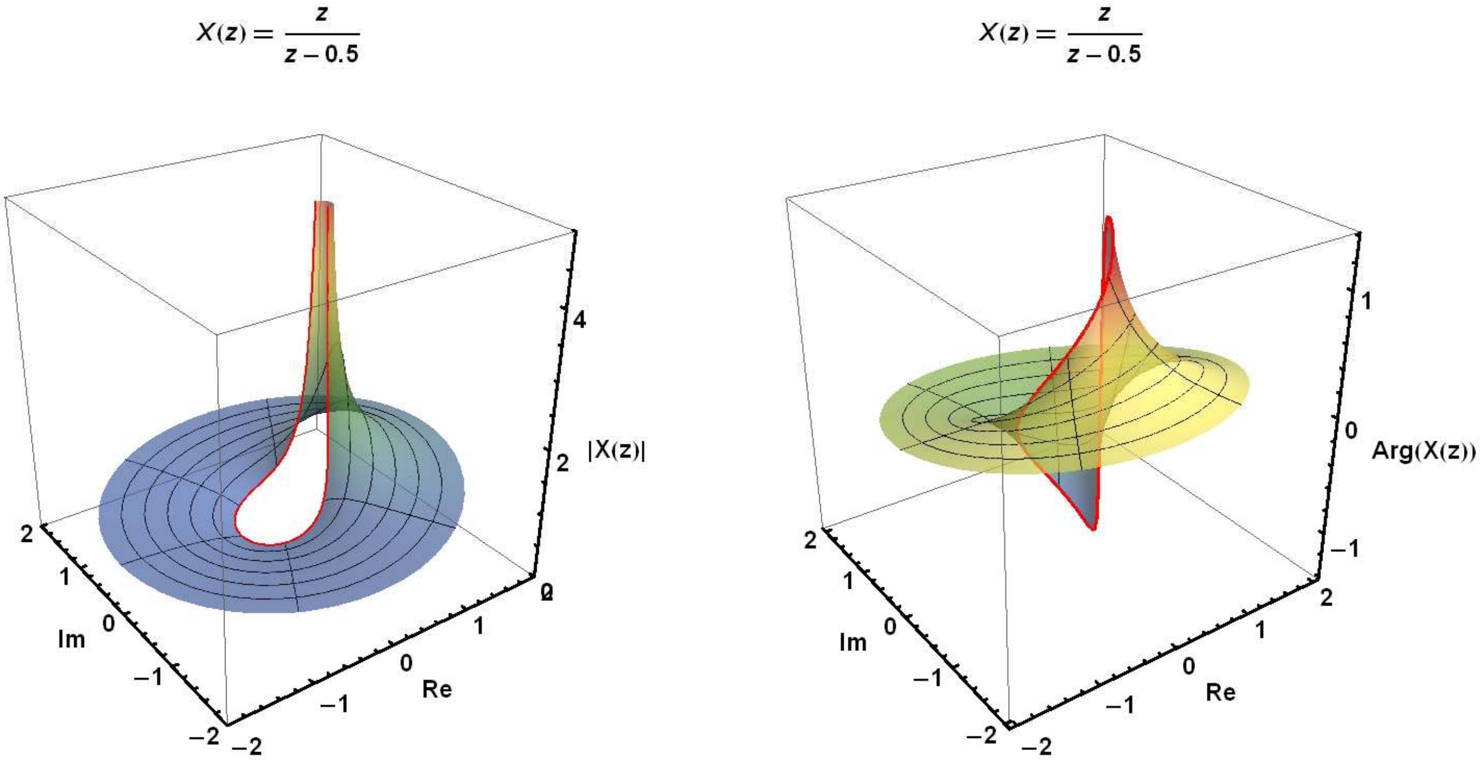
Z-transform of elementary DT signals

- Plot of function $X(z) = \frac{z}{z-a}$, red line$=\left\{X(z) : |z| = a\right\}$, $a = 0.5$



Z-transform of elementary DT signals

- Plot of function $x(n)=a^n u(n) \quad Z\{x(n)\}=\frac{z}{z-a}, \quad \text{RoC}=\{z:|z|>|a|\}$



Z-transform of elementary DT signals

- z-transform of sine and cosine functions

$$x(n)=\cos(\omega_0 n)u(n), \quad \text{RoC=}\{z:|z|>1\}$$

$$Z\{x(n)\}=\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}=\frac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$$

$$x(n)=\sin(\omega_0 n)u(n), \quad \text{RoC=}\{z:|z|>1\}$$

$$Z\{x(n)\}=\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}=\frac{z\sin(\omega_0)}{z^2-2z\cos(\omega_0)+1}$$

Z-transform of elementary DT signals

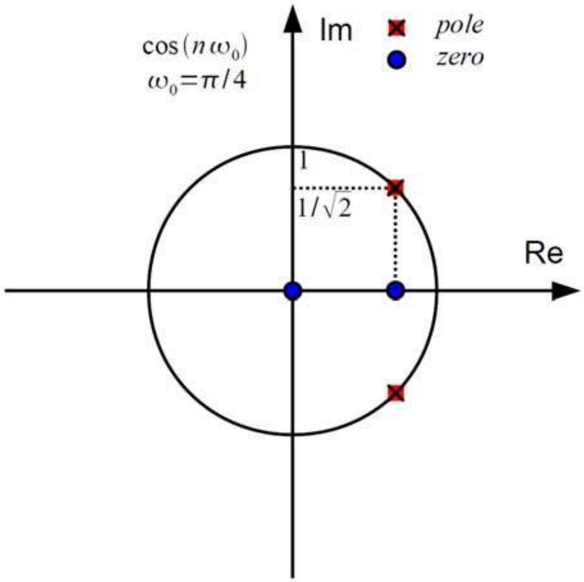
- Poles and zeroes sine and cosine functions in the z-domain

poles and zeroes:

$$z^2-2z\cos(\omega_0)+1=0 \Rightarrow \begin{cases} p_1=\cos(\omega_0)+i\left|\sin(\omega_0)\right| \\ p_2=\cos(\omega_0)-i\left|\sin(\omega_0)\right| \end{cases}$$

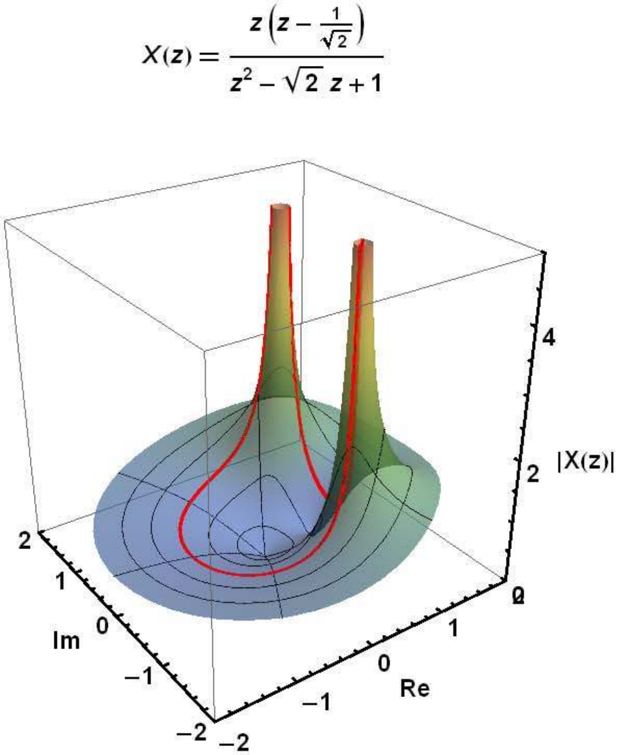
$$z^2-z\cos(\omega_0)=0 \quad \Rightarrow \begin{cases} z_1=0 \\ z_2=\cos(\omega_0) \end{cases}$$

$$z\sin(\omega_0)=0 \quad \Rightarrow \left\{ z_1=0 \right.$$

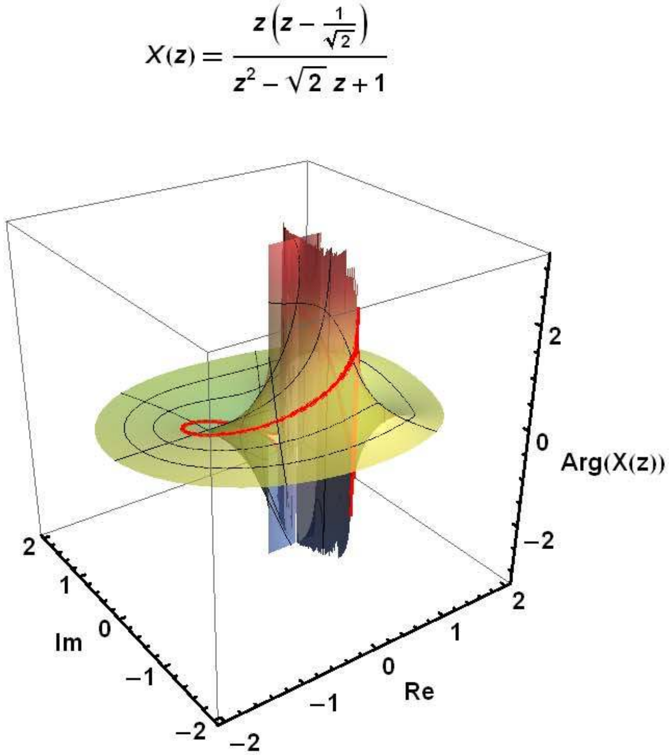


Z-transform of elementary DT signals

- Plot of function $X(z) = \dfrac{z^2 - z\cos(\omega_0)}{z^2 - 2z\cos(\omega_0) + 1}$, red line=$\{X(z) : |z| = 1\}$, $\omega_0 = \pi/4$



$$X(z) = \frac{z\left(z - \frac{1}{\sqrt{2}}\right)}{z^2 - \sqrt{2}\,z + 1}$$

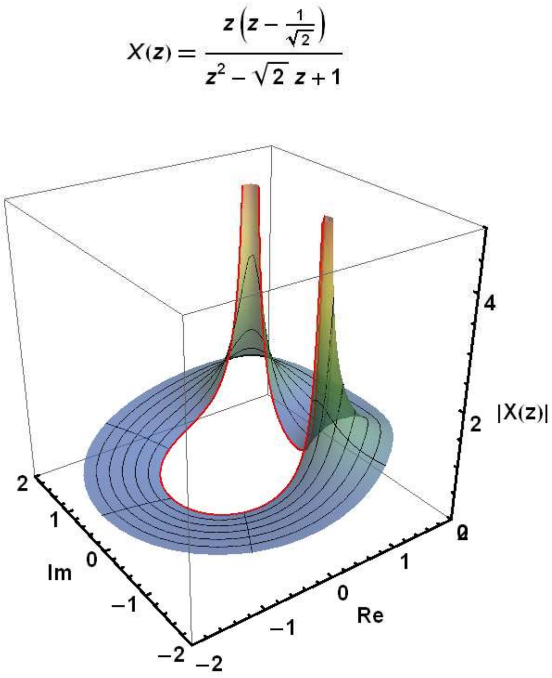


$$X(z) = \frac{z\left(z - \frac{1}{\sqrt{2}}\right)}{z^2 - \sqrt{2}\,z + 1}$$

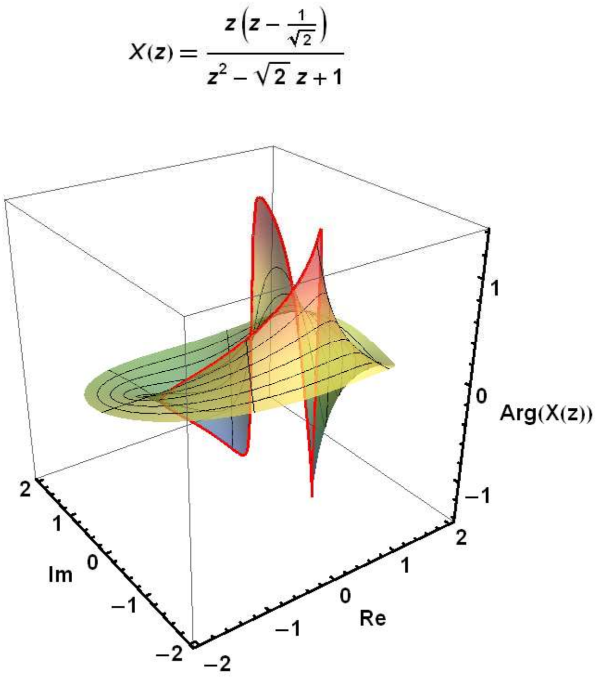
Z-transform of elementary DT signals

- Plot of function $x(n)=\cos(n\omega_0)u(n) \quad Z\{x(n)\}=\dfrac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1},$

$$\text{RoC}=\{z:|z|>1\},\omega_0=\pi/4$$



$$X(z)=\frac{z\left(z-\frac{1}{\sqrt{2}}\right)}{z^2-\sqrt{2}\,z+1}$$



$$X(z)=\frac{z\left(z-\frac{1}{\sqrt{2}}\right)}{z^2-\sqrt{2}\,z+1}$$

Operations of DT systems – basic operations

name	time	z-domain
addition	$y(n)=g(n)+h(n)$	$Y(z)=G(z)+H(z)$
multiplication with constant	$y(n)=a\cdot g(n)$	$Y(z)=aG(z)$
multiplication	$y(n)=g(n)\cdot h(n)$	$Y(z)=\oint_C G(v)H\left(\frac{z}{v}\right)v^{-1}dv$
time shift	$y(n)=S^k\{g(n)\}=g(n-k)$	$Y(z)=z^{-k}G(z)$
accumulation	$y(n)=\sum_{k=-\infty}^{n} g(k)$	$Y(z)=\frac{1}{1-z^{-1}}G(z)$
DT convolution	$y(n)=g(n)*h(n)$	$Y(z)=G(z)\cdot H(z)$

Operations of DT systems – basic operations

Addition, multiplication with constant comes from the linearity property.

Time shift operation:

$$Z\left\{x\left(n-k\right)\right\}=\sum_{n=-\infty}^{\infty} z^{-n}x\left(n-k\right)\underset{m=n-k}{=}\sum_{m=-\infty}^{\infty} z^{-m-k}x\left(m\right)=z^{-k}\sum_{m=-\infty}^{\infty} z^{-m}x\left(m\right)=z^{-k}X\left(z\right)$$

DT Convolution:

$$Z\left\{x_1\left(n\right)*x_2\left(n\right)\right\}=Z\left\{\sum_{k=-\infty}^{\infty} x_1\left(k\right)\cdot x_2\left(n-k\right)\right\}=\sum_{n=-\infty}^{\infty} z^{-n}\sum_{k=-\infty}^{\infty} x_1\left(k\right)\cdot x_2\left(n-k\right)=$$

$$=\sum_{k=-\infty}^{\infty} x_1\left(k\right)\cdot\sum_{n=-\infty}^{\infty} x_2\left(n-k\right)z^{-n}=\sum_{k=-\infty}^{\infty} x_1\left(k\right)\cdot z^{-k}\sum_{m=-\infty}^{\infty} x_2\left(m\right)z^{-m}=$$

$$=X_1\left(z\right)\cdot X_2\left(z\right)$$

Operations of DT systems – basic operations

Accumulation

$$y(n)=\sum_{k=-\infty}^{n} x(k) \text{ using the time shift property}$$

$$Y(z)=\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{n} x(k)z^{-n} = \sum_{n=-\infty}^{\infty}\left(x(n)+x(n-1)+\cdots+x(-\infty)\right)\cdot z^{-n} =$$

$$= X(z)\left(1+z^{-1}+\cdots+z^{-\infty}\right)= X(z)\sum_{k=0}^{\infty} z^{-k} = X(z)\frac{1}{1-z^{-1}}$$

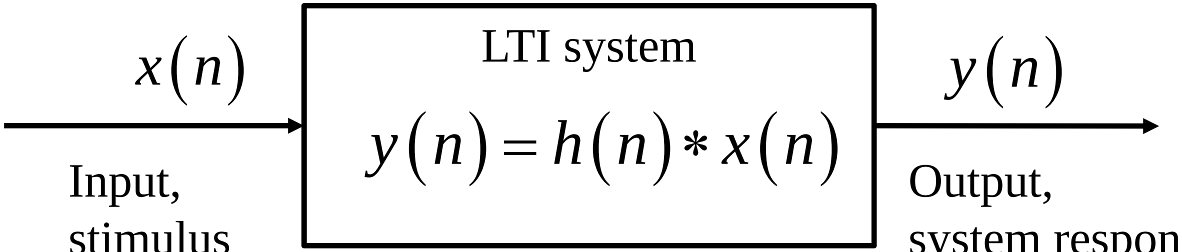
alternatively accumulation can be written as a convolution

$$y(n)=\sum_{k=-\infty}^{n} x(k)=\sum_{k=-\infty}^{n} x(k)u(n-k)=\sum_{k=-\infty}^{\infty} x(k)u(n-k)=x(n)*u(n)$$

$$Y(z)=X(z)\cdot U(z)=X(z)\frac{1}{1-z^{-1}}$$

LTI systems in the z-domain

A LTI system in the time domain performs a discrete time convolution on its input with its impulse response function



$x(n)$

LTI system

$$y(n)=h(n)*x(n)$$

$y(n)$

Input,
stimulus

Output,
system response

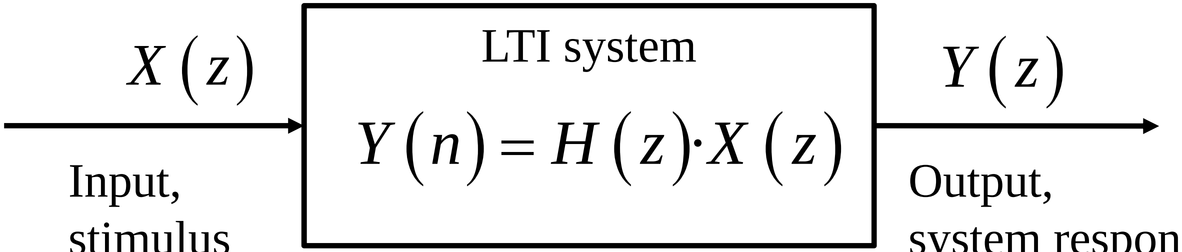
An LTI system is fully characterized by its impulse response function (*h*(*n*)).

If we examine the same system in the transformed domain, we can use the convolution to multiplication property of the z-transform:

$$x_1(n)*x_2(n)\overset{z}{\rightleftharpoons}X_1(z)\cdot X_2(z)$$

LTI systems in the z-domain

A LTI system in the z-domain performs a multiplication on its input’s z-transform and the system’s transfer function



An LTI system is fully characterized by its transfer function *H*(*z*).

$$\left.\begin{array}{l} x(n)\xrightleftharpoons{Z}X(z) \\ y(n)\xrightleftharpoons{Z}Y(z)=H(z)X(z) \\ h(n)\xrightleftharpoons{Z}H(z) \end{array}\right\} \quad \begin{array}{l} \text{if } x(n)=\delta(n)\xrightleftharpoons{Z}1 \\ y(n)=h(n)\xrightleftharpoons{Z}H(z) \end{array}$$

Transfer function of a system

- Remember the linear constant-coefficient difference equation which described an arbitrary LTI system:

$$\sum_{k=0}^{N} a_k \cdot y(n-k) = \sum_{k=0}^{M} b_k \cdot x(n-k)$$

Z-transforming it (using the time shift property and linearity):

$$Z\{f(n-k)\} = z^{-k} F(z), \quad Z\{c_1 f_1(n) + c_2 f_2(n)\} = c_1 F_1(z) + c_2 F_2(z)$$

$$Z\{y(n)\} = Y(z), \quad Z\{x(n)\} = X(z),$$

$$Z\left\{\sum_{k=0}^{N} a_k \cdot y(n-k)\right\} = Z\left\{\sum_{k=0}^{M} b_k \cdot x(n-k)\right\}$$

$$\sum_{k=0}^{N} a_k \cdot z^{-k} Y(z) = \sum_{k=0}^{M} b_k \cdot z^{-k} X(z)$$

Transfer function of a system

- We can define the transfer function of an LTI system:

$$\sum_{k=0}^{N} a_k \cdot z^{-k} Y(z) = \sum_{k=0}^{M} b_k \cdot z^{-k} X(z)$$

$$Y(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{\sum_{k=0}^{N} a_k \cdot z^{-k}} X(z) = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2} + \ldots + b_M \cdot z^{-M}}{a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} + \ldots + a_N \cdot z^{-N}} X(z)$$

if $x(n) = \delta(n) \rightarrow y(n) = h(n)$ impulse response function

$X(z) = 1 \rightarrow Y(z) = H(z)$ transfer function

$$H(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{\sum_{k=0}^{N} a_k \cdot z^{-k}} = \frac{B(z)}{A(z)}, \quad B(z), A(z) \text{ polynomials}$$

Transfer function of a system – poles and zeroes

- We define the poles and zeroes of an LTI system as the poles and zeroes of it's transfer function:

$$H(z)=\frac{b_0+b_1\cdot z^{-1}+b_2\cdot z^{-2}+\ldots+b_M\cdot z^{-M}}{a_0+a_1\cdot z^{-1}+a_2\cdot z^{-2}+\ldots+a_N\cdot z^{-N}}=\frac{B(z)}{A(z)},\quad B(z),A(z)\text{ polynomials}$$

$$\text{roots of } B(z)=0,\quad o_1,o_2,\ldots,o_M \text{ are called zeroes}$$

$$\text{roots of } \mathrm{A}(z)=0,\quad p_1,p_2,\ldots,p_M \text{ are called poles}$$

$$H(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0}{a_0}\frac{(z-o_1)(z-o_2)\cdots(z-o_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}=z^{N-M}\frac{b_0}{a_0}\frac{\prod_{m=1}^{M}(z-o_m)}{\prod_{n=1}^{N}(z-p_n)},\quad o_i,p_i\in\mathbb{C}$$

- For understanding what does a pole or a zero “do” we need to understand the connection between the z-transform and the discrete Fourier transform.

Z-transform and the Fourier transform

- Remember the definition of the bilateral z-transform:

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}, \quad z\in\mathbb{C}$$

- If we substitute $z=e^{j\cdot\omega}, \omega\in[-\pi,\pi]$

  we get the Fourier series of *X* where the Fourier coefficients are the time samples.

- Note that usually we are interested in a periodic, continuous time function’s decomposition into Fourier series, where the coefficients represent the weight of the frequency component, but from the duality property we can do this in the opposite direction.

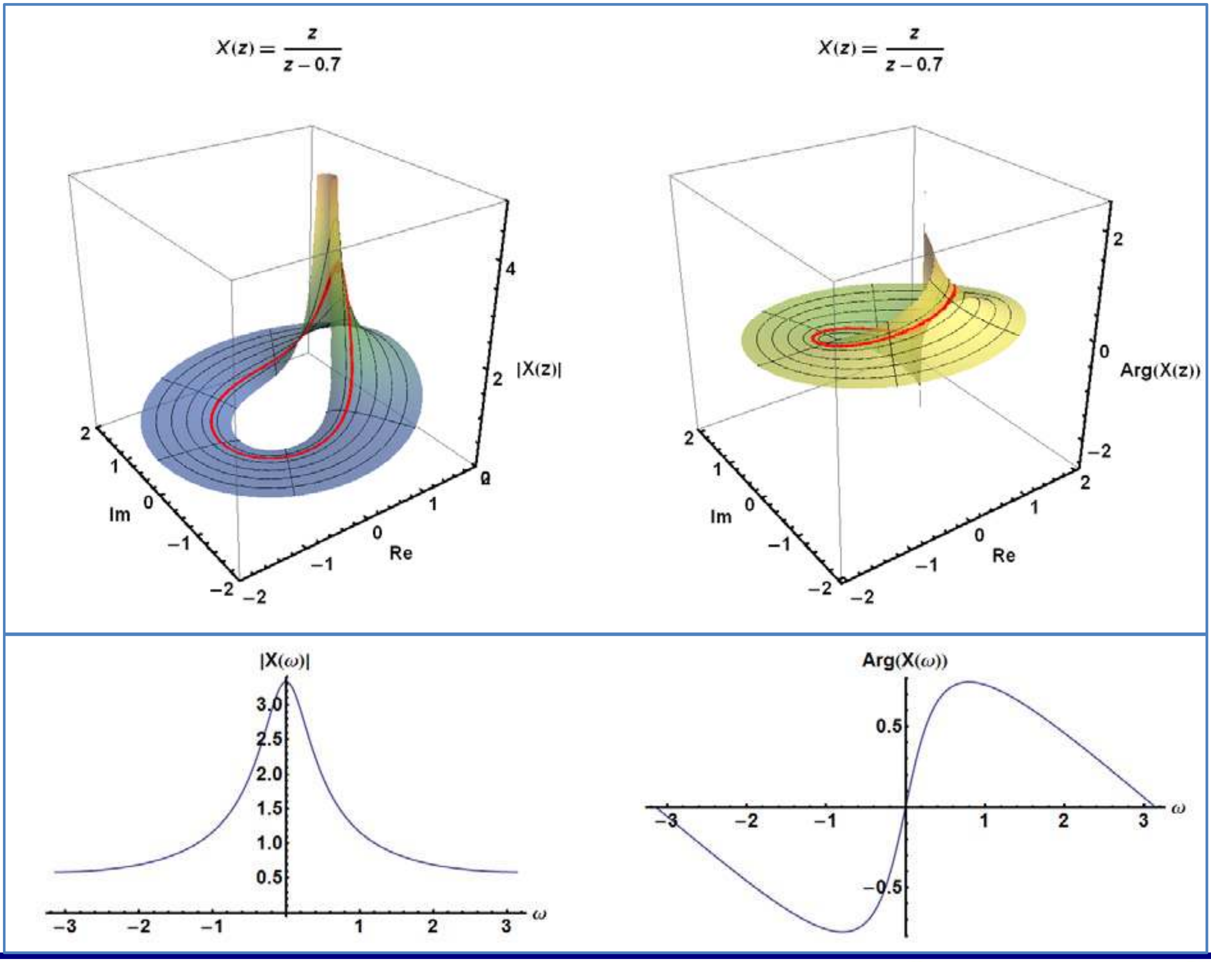
Z-transform and the Fourier transform

- Here the frequency function is continuous and periodic, and the Fourier coefficients are the time samples.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j \cdot \omega \cdot n}, \quad \omega \in [-\pi, \pi]$$

- *X* is a function of the angular frequency. We view the original whole complex z plane only on the unit circle.
- The following illustration shows the connection.
- On the upper figure the red line shows the function values on the unit circle. On the lower figure the *x* axis, the angular frequency is the “unit circle”, folded out to a line.

Z-transform and the Fourier transform



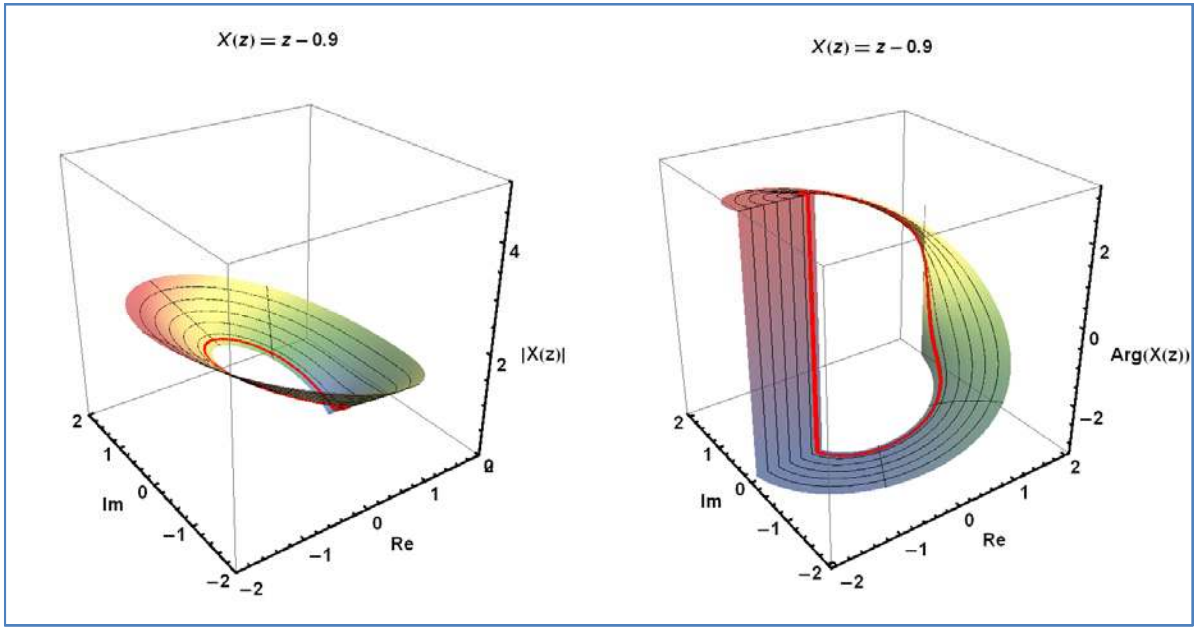
Transfer function of a system – poles and zeroes

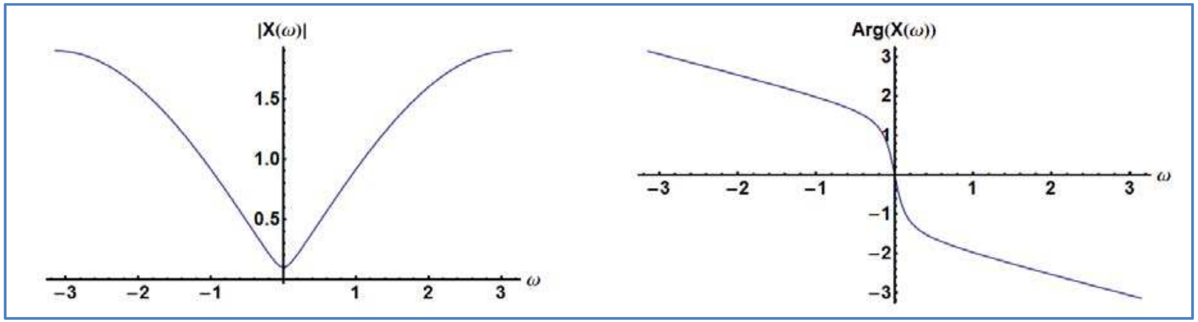
- If we view the system in the frequency domain (properly introduced later) we can say that a zero in the transfer function

  $$o_i = r \cdot e^{j \cdot \omega}, r \in \mathbb{R}, \omega \in [-\pi, \pi]$$

  at angular frequency omega is attenuating that frequency and it’s surroundings.
- The attenuation strength is dependent how close *r* is to 1 (how close is the zero to the unit circle)
- The next figure illustrates a simple zero and it’s frequency characteristics with zero at: $o_1 = 0.9 \cdot e^{j \cdot 0}$

Transfer function of a system – poles and zeroes





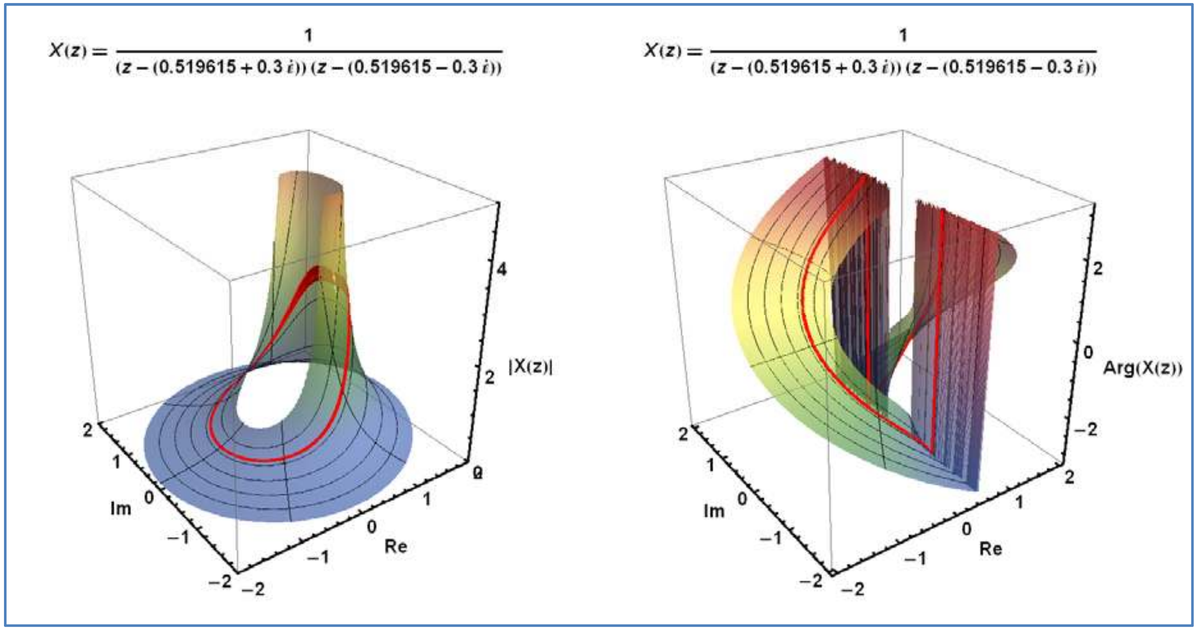
Transfer function of a system – poles and zeroes

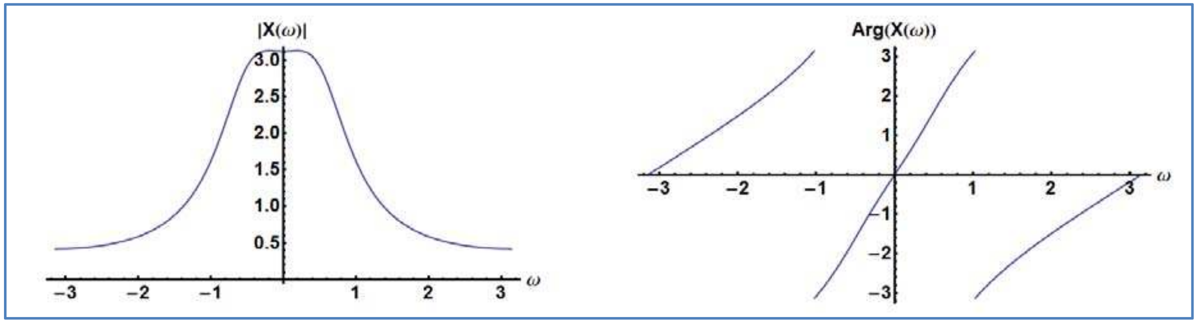
- If we view the system in the frequency domain a pole in the transfer function $p_i = r \cdot e^{j \cdot \omega}, r \in \mathbb{R}, \omega \in [-\pi, \pi]$

  at an angular frequency omega is amplifying that frequency and it’s surroundings.
- The amplifying strength is dependent how close *r* is to 1 (how close is the pole to the unit circle)
- The next figure illustrates a simple two pole and it’s frequency characteristics with poles:

$$p_1 = 0.6 \cdot e^{j \cdot 2\pi / 12}, p_2 = 0.6 \cdot e^{-j \cdot 2\pi / 12}$$

Transfer function of a system – poles and zeroes





Transfer function of a system

- It can be useful to have the transfer function in the following form:

$$H(z) = Q(z^{-1}) + \sum_{i=1}^{N} \frac{c_i z}{z - p_i}, \quad Q(z^{-1}) \text{ polynomial or 0, } c_i \text{ constants, } p_i \text{ poles of } H(z)$$

It will be useful to perform the inverse z-transform because

$$Z^{-1}\left\{q_k z^{-k}\right\} = q_k \delta(n-k)$$

$$Z^{-1}\left\{Q(z^{-1})\right\} = Z^{-1}\left\{\sum_{k=0}^{\deg(Q(z^{-1}))} q_k z^{-k}\right\} = \sum_{k=0}^{\deg(Q(z^{-1}))} q_k \delta(n-k)$$

Remember the Kronecker delta correction at the solution in the time domain

Transfer function of a system

- And for the root factored term

$$Z^{-1}\left\{\frac{c_i z}{z-p_i}\right\}=c_i\cdot p_i^n\cdot u(n)$$

$$Z^{-1}\left\{\sum_{i=1}^{N}\frac{c_i z}{z-p_i}\right\}=\sum_{i=1}^{N}c_i\cdot p_i^n\cdot u(n)$$

Remember the response for the particular solution was given as

$$y_H(n)=\sum_{l=1}^{N}C_l\lambda_l^n \quad \text{so } C_l=c_i, \quad \lambda_l^n=p_i^n$$

So we can compute the inverse z-transform easily. The roots of the characteristic polynomial are the poles of the transfer function.

Review of polynomial manipulation

- We can manipulate polynomials like:

Let $A\left(z\right)$ a polynomial

it's standard form is: $A\left(z\right)=\sum_{k=0}^{N}a_k\cdot z^k=a_0+a_1z^1+a_2z^2+\cdots+a_Nz^N$

from the fundamental theorem of algebra we know that $A\left(z\right)$ has exactly N roots.

Let us denote them as $r_1,r_2,\ldots,r_N$ note that $r_i=r_j$ can happen (multiplicative roots)

if we know the roots, we can write the root factored form

$$A\left(z\right)=\left(z-r_1\right)\left(z-r_2\right)\cdots\left(z-r_N\right)=\prod_{i=1}^{N}\left(z-r_i\right)$$

from the complex conjugate root theorem we also know that if $a_i\in\mathbb{R}\forall i$

then the roots appear as complex conjugate pairs $r_i=a+bj,r_j=a-bj$

E.g. $x^2+1=0\rightarrow0\pm1j=\pm j,\qquad x^2-1=0\rightarrow0\pm j\cdot j=\pm1,$

Review of polynomial long division

- Polynomials of lower degree can divide a polynomial with a higher degree. We will use this to analyze LTI systems.
- Note that you learned this method in elementary school to divide numbers.

Let $\mathrm{B}(z)$ a polynomial of degree M, and $A(z)$ a polynomial of degree $N, M \geq N$

we want to have $\mathrm{H}(z)=\frac{B(z)}{A(z)}=Q(z)+\frac{R(z)}{A(z)}, \deg(R(z))<\deg(A(z))$

$Q(z)$ is the quotient, $R(z)$ is the remainder

- The method will be illustrated on an example.

Review of polynomial long division

$$\mathrm{H}(z)=\frac{B(z)}{A(z)}=Q(z)+\frac{R(z)}{A(z)},\quad H(z)=\frac{z^5+4z^4+8z^3+9z^2+3z+3}{z^2+2z+4}$$

$$\left(z^5+4z^4+8z^3+9x^2+3x+3\right):\left(z^2+2z+4\right)=$$

first divide the highest degree term in the divident with the highest term in the divisor to get the first term of the quotient

$$z^5/z^2=z^3$$

then multiply back the result with the divisor to get the "error" we made

$$z^3\cdot\left(z^2+2z+4\right)=z^5+2z^4+4z^3$$

substract the error term to lower the degree of the divident

$$z^5+4z^4+\ 8z^3+\ 9z^2+\ 3z+\ 3:\left(z^2+2z+4\right)=z^3$$

$$-(\ z^5+2z^4+\ 4z^3)=$$

$$=\qquad 2z^4+4z^3+\ 9z^2+3z+\ \ 3$$

Review of polynomial long division

continue from step 1 with the newly obtained polynomial until it's degree is smaller than the divisor's

$$
\begin{array}{lllllll}
 & & 2z^4 + 4z^3 + 9z^2 + & 3z + & 3 : \left(z^2+2z+4\right) = 2z^2 \\
 & -(& 2z^4 + 4z^3 + 8z^2) = & & \\
= & & z^2 + & 3z + & 3 : \left(z^2+2z+4\right) = 1 \\
 & & -(\quad z^2 + & 2z + & 4) = \\
= & & & z - & 1 = R\left(z\right)
\end{array}
$$

$$Q\left(z\right) = z^3 + 2z^2 + 1$$

$$\mathrm{H}\left(z\right) = \frac{B\left(z\right)}{A\left(z\right)} = Q\left(z\right) + \frac{R\left(z\right)}{A\left(z\right)},$$

$$H\left(z\right) = \frac{z^5 + 4z^4 + 8z^3 + 9z^2 + 3z + 3}{z^2 + 2z + 4} = (z^3 + 2z^2 + 1) + \frac{z-1}{z^2+2z+4}$$

Review of partial fraction expansion of polynomials

Given two polynomials $A(z), R(z)$ with degrees $\deg(A(z)) > \deg(R(z))$

partial fraction expansion can be obtained by assuming that

$$\frac{R(z)}{A(z)} = \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \frac{c_3}{z-p_3} + \cdots + \frac{c_N}{z-p_N}, c_i, p_i \in \mathbb{C}, \text{where } p_i \text{ are the roots of } A(z)$$

if p_i has multiplicity r then the corresponding terms will be

$$\frac{R(z)}{A(z)} = \frac{R(z)}{(z-p_i)^r \tilde{A}(z)} = \frac{c_1}{z-p_i} + \frac{c_2}{(z-p_i)^2} + \frac{c_3}{(z-p_i)^3} + \frac{c_r}{(z-p_i)^r} + \frac{c_{r+1}}{z-p_{i+1}} + \cdots + \frac{c_N}{z-p_N}$$

we can find the coefficients c_i with multiplying back for the common denominator $A(z)$

and for the numerator back substituting each $z = p_i$

The method will be illustrated by an example

Review of partial fraction expansion of polynomials

Let's use the polynomials from the long division example:

$$\frac{R(z)}{A(z)}=\frac{z-1}{z^2+2z+4}=\frac{z-1}{\left[z-\left(-1+\sqrt{3}j\right)\right]\left[z-\left(-1-\sqrt{3}j\right)\right]}=\frac{A}{z-\left(-1+\sqrt{3}j\right)}+\frac{B}{z-\left(-1-\sqrt{3}j\right)}$$

multiply back for the common divisor

$$\frac{z-1}{\left[z-\left(-1+\sqrt{3}j\right)\right]\left[z-\left(-1-\sqrt{3}j\right)\right]}=\frac{A\left[z-\left(-1-\sqrt{3}j\right)\right]+B\left[z-\left(-1+\sqrt{3}j\right)\right]}{\left[z-\left(-1+\sqrt{3}j\right)\right]\left[z-\left(-1-\sqrt{3}j\right)\right]}$$

equate the numerators

$$z-1=A\left[z-\left(-1-\sqrt{3}j\right)\right]+B\left[z-\left(-1+\sqrt{3}j\right)\right]=A\cdot z-A\left(-1-\sqrt{3}j\right)+B\cdot z-B\left(-1+\sqrt{3}j\right)$$

solve the equation system for the appropriate terms with the appropriate power of *z*

$$\left.\begin{aligned} z&=A\cdot z+B\cdot z\\ -1&=-A\left(-1-\sqrt{3}j\right)-B\left(-1+\sqrt{3}j\right)\end{aligned}\right\}A\rightarrow\frac{1}{2}+\frac{j}{\sqrt{3}},B\rightarrow\frac{1}{2}-\frac{j}{\sqrt{3}}$$

Review of partial fraction expansion of polynomials

Another example:

$$\frac{R(z)}{A(z)}=\frac{z^3-z-1}{(z-1)(z+2)(z-3)^2}=\frac{A}{z-1}+\frac{B}{z+2}+\frac{C}{z+3}+\frac{D}{(z+3)^2}$$

multiply back for the common divisor

$$\frac{z^3-z-1}{(z-1)(z+2)(z-3)^2}=\frac{A(z+2)(z+3)^2+B(z-1)(z+3)^2+C(z-1)(z+2)(z+3)+D(z-1)(z+2)}{(z-1)(z+2)(z-3)^2}$$

equate the numerators

$$z^3-z-1=A(z+2)(z+3)^2+B(z-1)(z+3)^2+C(z-1)(z+2)(z+3)+D(z-1)(z+2)$$

$$z^3-z-1=(A+B+C)z^3+(8A+5B+4C+D)z^2+(21A+3B+C+D)z+18A-9B-6C-2D$$

Review of partial fraction expansion of polynomials

solve the equation system for the appropriate terms with the appropriate power of *z*

$$\left.\begin{aligned} z^3 &= (A+B+C)z^3 \\ 0\cdot z^2 &= (8A+5B+4C+D)z^2 \\ -z &= (21A+3B+C+D)z \\ -1 &= 18A-9B-6C-2D \end{aligned}\right\} A \to -\frac{1}{48}, B \to \frac{7}{3}, C \to -\frac{21}{16}, D \to -\frac{25}{4}$$

$$\frac{R(z)}{A(z)} = \frac{z^3-z-1}{(z-1)(z+2)(z-3)^2} = -\frac{1}{48(z-1)} + \frac{7}{3(z+2)} - \frac{21}{16(z+3)} - \frac{25}{4(z+3)^2}$$

LTI systems – BIBO stability review

An LTI system is BIBO stable iff the system's impulse response is absolute summable.

If a $y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$ system is BIBO stable,

$S_h = \sum_{k=-\infty}^{\infty} |h(k)| < \infty$ must hold.

For a FIR system this always holds.

For an IIR system the impulse response function is given as:

$$h(n) = \sum_{l=1}^{N} C_l \lambda_l^n + \underbrace{\sum_{i=0}^{\max(M-N,0)} w_i \delta(n-i)}_{<\infty}$$

$S_h < \infty$ holds only if $|\lambda_i| < 1$.

LTI systems – BIBO stability analysis in z-domain

- If we know the poles of the transfer function we can immediately see if the system is BIBO stable or not.
- If the all the poles are within the unit circle the system is BIBO stable.

$$h(n) = \sum_{l=1}^{N} C_l \lambda_l^n + \underbrace{\sum_{i=0}^{\max(M-N,0)} w_i \delta(n-i)}_{<\infty}$$

$S_h < \infty$ holds only if $\lambda_i < 1$

and we saw that $\lambda_i = p_i$ poles of the transfer function.

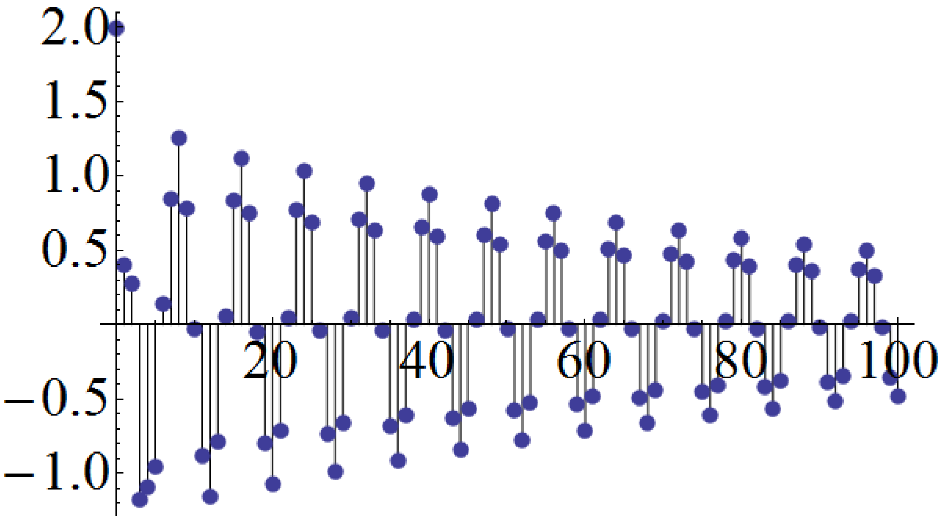
- If a pole is outside or on the unit circle the system is not BIBO stable.

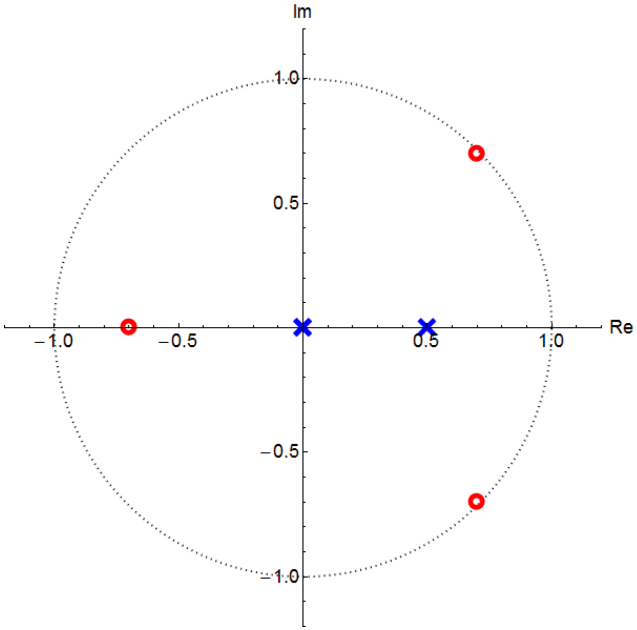
LTI systems – BIBO stability analysis in z-domain

- Example: $y(n) - 0.7y(n-1) + 0.686y(n-3) = 2x(n) - x(n-1)$

$$H(z) = \frac{2 - z^{-1}}{1 - 0.7z^{-1} + 0.686z^{-3}} = \frac{2z^3 - z^2}{z^3 - 0.7z^2 + 0.686}$$

poles: $p_1 = \text{-}0.7, \quad p_2 = 0.7 + 0.7j \quad p_3 = 0.7 + 0.7j$



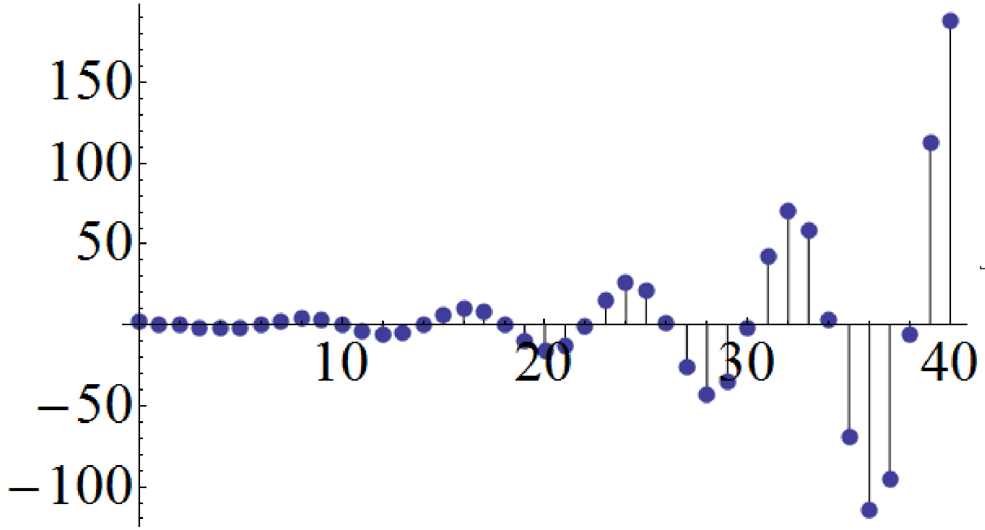


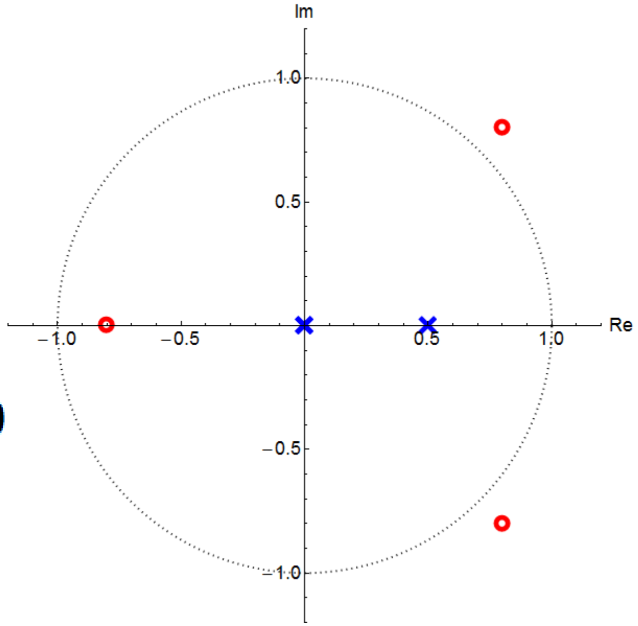
LTI systems – BIBO stability analysis in z-domain

- Example: $y(n) - 0.8y(n-1) + 1.024y(n-3) = 2x(n) - x(n-1)$

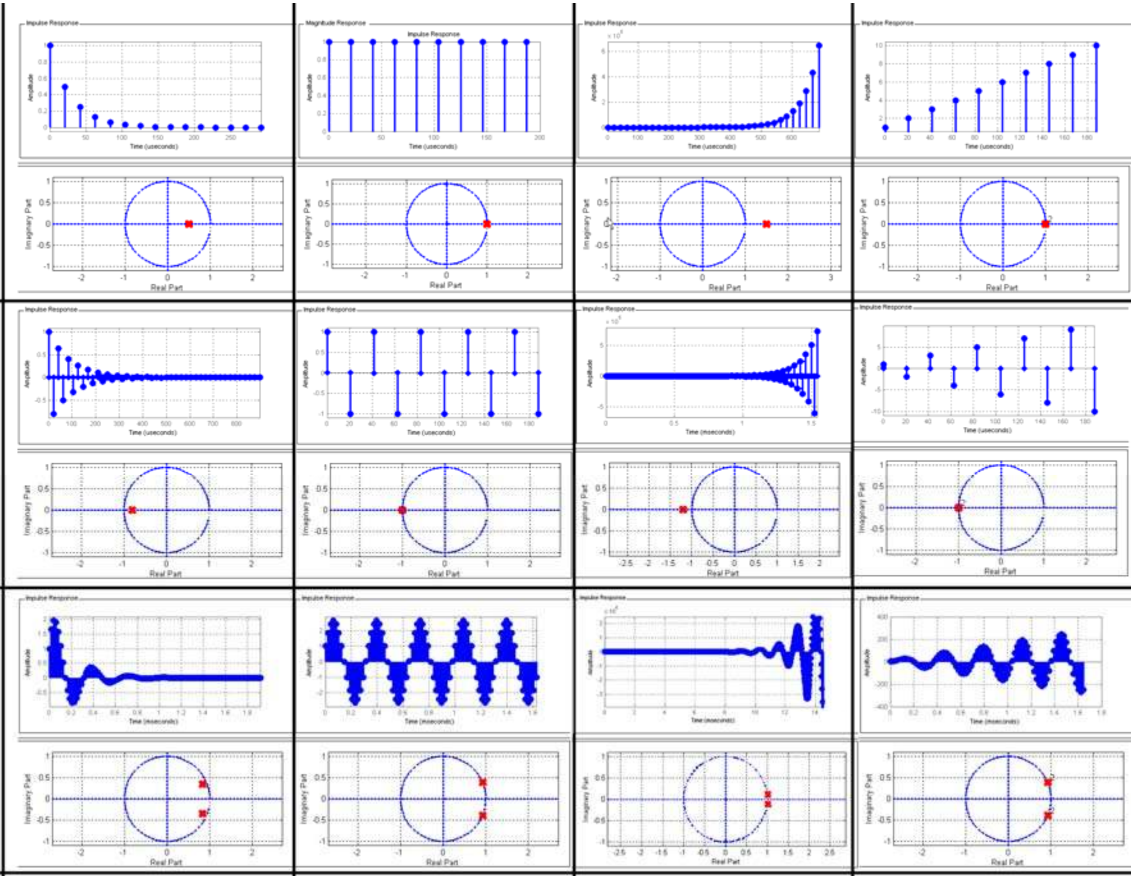
$$H(z) = \frac{2 - z^{-1}}{1 - 0.8z^{-1} + 1.024z^{-3}} = \frac{2z^3 - z^2}{z^3 - 0.8z^2 + 1.024}$$

poles: $p_1 = \text{-}0.8, \quad p_2 = 0.8 + 0.8j \quad p_3 = 0.8 + 0.8j$





Impulse response characteristics based on poles



$p \in \mathbb{R}$
$p > 0$

$p \in \mathbb{R}$
$p < 0$

$p_1, p_2 \in \mathbb{C}$
$p_1 = p_2^*$

|p|<1

|p|=1

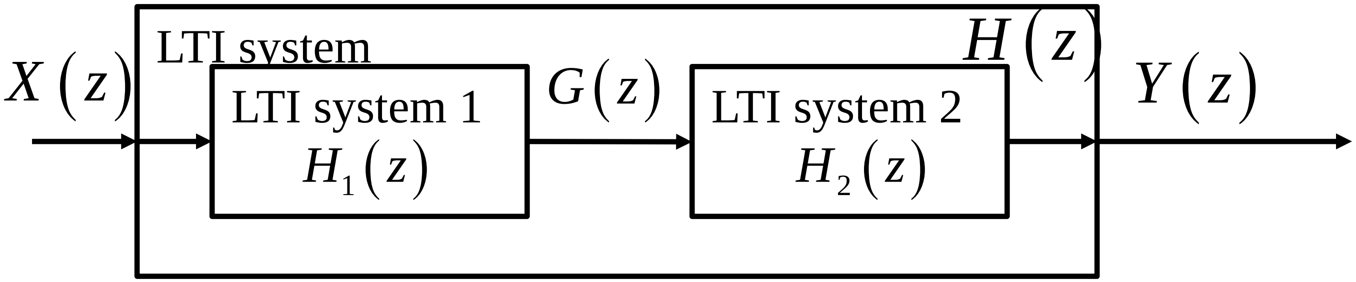
|p|>1

|p|=1, multiple

Convolution properties in the z-domain

- Associativity – serial combination of LTI systems

$$\left.\begin{aligned}
y(n) &= h(n) * x(n)\\
y(n) &= \underbrace{\left(h_2(n) * h_1(n)\right)}_{h(n)} * x(n)\\
y(n) &= h_2(n) * \underbrace{\left(h_1(n) * x(n)\right)}_{g(n)}\\
h(n) &= h_1(n) * h_2(n)
\end{aligned}\right\} \overset{Z}{\rightleftharpoons} \left\{\begin{aligned}
Y(z) &= H(z) \cdot X(z)\\
Y(z) &= \underbrace{\left(H_2(z) \cdot H_1(z)\right)}_{H(z)} \cdot X(z)\\
Y(z) &= H_2(z) \cdot \underbrace{\left(H_1(z) \cdot X(z)\right)}_{G(z)}\\
H(z) &= H_1(z) \cdot H_2(z)
\end{aligned}\right.$$

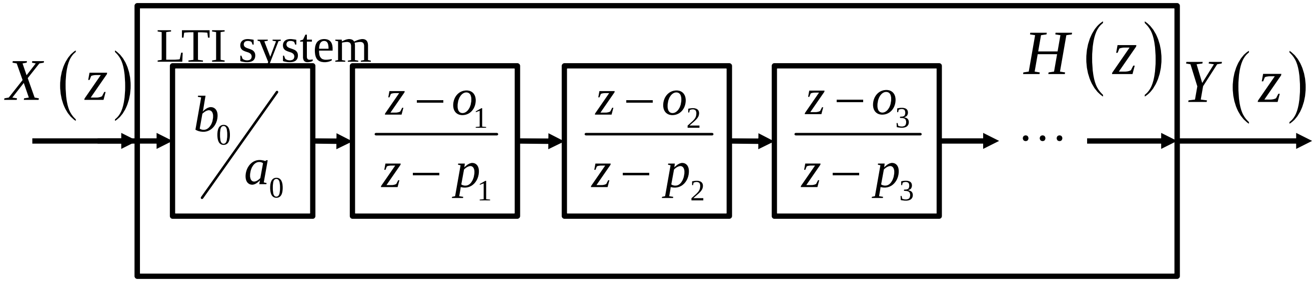


Serial combination of simple IIR systems

- If we write the system as

$$H(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0}{a_0}\frac{(z-o_1)(z-o_2)\cdots(z-o_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}=z^{N-M}\frac{b_0}{a_0}\frac{\prod_{m=1}^{M}(z-o_m)}{\prod_{n=1}^{N}(z-p_n)},\quad o_i,p_i\in\mathbb{C}$$

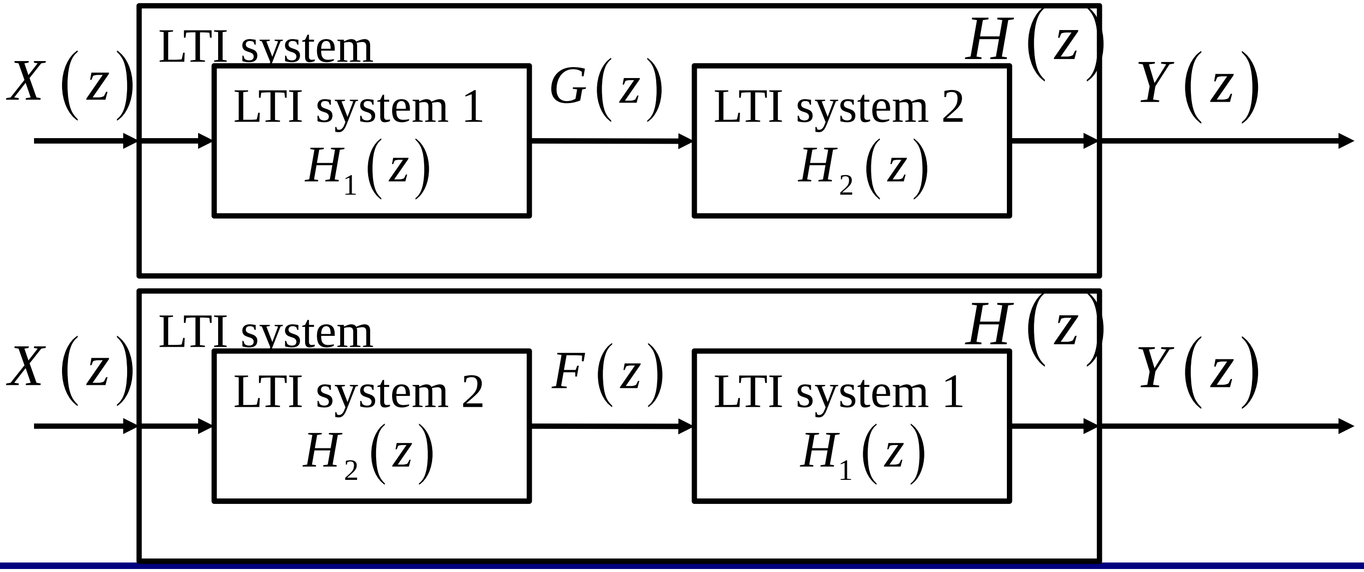
- It can be realized that simple one poles one zeroes can be cascaded to get the original transfer function.



Convolution properties in the z-domain

- Commutativity – switch ability of LTI systems

$$\left.\begin{aligned} y(n) &= \left(h_1(n) * h_2(n)\right) * x(n) \\ y(n) &= \left(h_2(n) * h_1(n)\right) * x(n) \\ h(n) &= h_1(n) * h_2(n) = h_2(n) * h_1(n) \end{aligned}\right\} \overset{Z}{\rightleftharpoons} \left\{\begin{aligned} Y(z) &= \left(H_1(z) \cdot H_2(z)\right) \cdot X(z) \\ Y(z) &= \left(H_2(z) \cdot H_1(z)\right) \cdot X(z) \\ H(z) &= H_1(z) \cdot H_2(z) = H_2(z) \cdot H_1(z) \end{aligned}\right.$$



LTI system $H(z)$

$X(z)$ LTI system 1 $H_1(z)$ $G(z)$ LTI system 2 $H_2(z)$ $Y(z)$

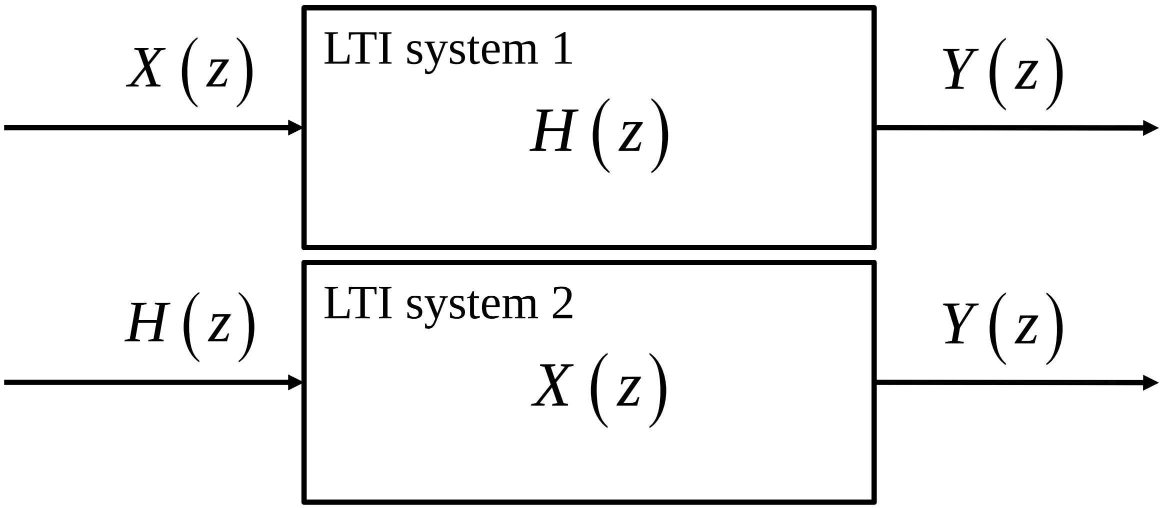
LTI system $H(z)$

$X(z)$ LTI system 2 $H_2(z)$ $F(z)$ LTI system 1 $H_1(z)$ $Y(z)$

Convolution properties in the z-domain

- Commutativity – switch ability of impulse response and excitation

$$\left.\begin{array}{l} y(n)=h(n)*x(n) \\ y(n)=x(n)*h(n) \end{array}\right\} \overset{z}{\rightleftharpoons} \left\{\begin{array}{l} Y(n)=H(z)\cdot X(z) \\ Y(n)=X(z)\cdot H(z) \end{array}\right.$$



$X(z)$

LTI system 1

$H(z)$

$Y(z)$

$H(z)$

LTI system 2

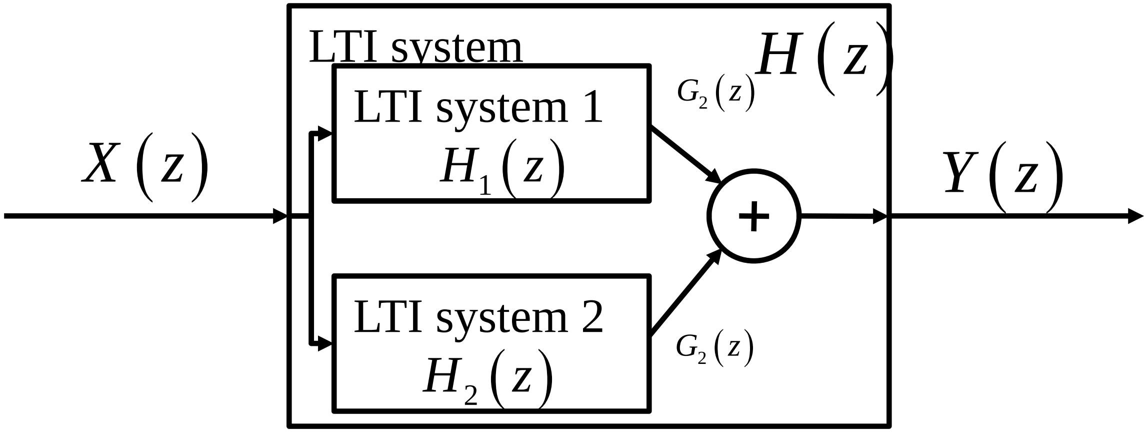
$X(z)$

$Y(z)$

Convolution properties in the z-domain

- Distributivity – parallel combination of LTI systems

$$\left.\begin{aligned} y(n)&=h(n)*x(n)\\ y(n)&=h_1(n)*x(n)+h_2(n)*x(n)\\ y(n)&=\left(h_1(n)+h_2(n)\right)*x(n)\\ h(n)&=h_1(n)+h_2(n)\end{aligned}\right\}\overset{z}{\rightleftarrows}\left\{\begin{aligned} Y(n)&=H(z)X(z)\\ Y(n)&=H_1(z)X(z)+H_2(z)X(z)\\ Y(n)&=\left(H_1(z)+H_2(z)\right)X(z)\\ H(n)&=H_1(z)+H_2(z)\end{aligned}\right.$$

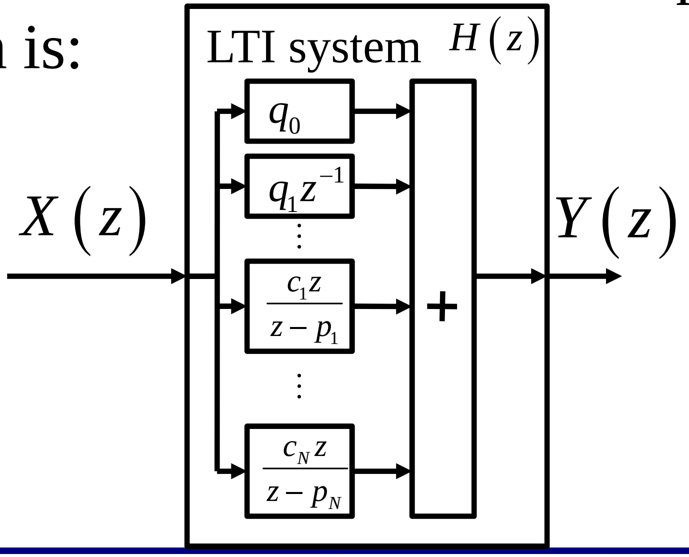


Parallel combinations of simple IIR systems

- If we write the system as

$$H(z)=Q\left(z^{-1}\right)+\sum_{i=1}^{N}\frac{c_i z}{z-p_i}=q_0+q_1z^{-1}+\cdots+\frac{c_1 z}{z-p_1}+\frac{c_2 z}{z-p_2}+\cdots+\frac{c_N z}{z-p_N}$$

which can be obtained by polynomial long division and partial fraction decomposition, it can be realized that a parallel implementation of the system is:



Linear phase filters

- A linear phase filter is a filter where the phase frequency response of the system is a linear function.

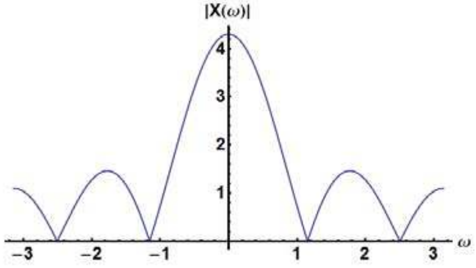
$$H\left(e^{j\cdot\omega}\right)=A(\omega)e^{j\cdot L(\omega)},\ \text{where } A(\omega):[-\pi,\pi]\mapsto\mathbb{R}\ \text{and } L(\omega)=\alpha\cdot\omega$$

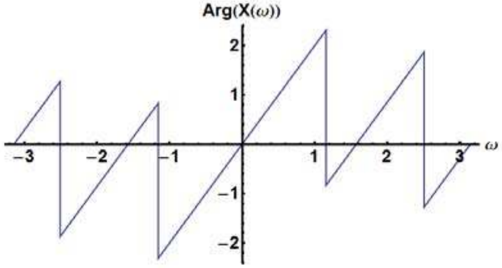
so the phase is a linear function of the angular frequency with slope α

- Such systems has their impulse response symmetric:

$$h(n)=h(N-1-n),n=0,1,\ldots,N-1$$

- E.g. $h(n)=\{\underset{\uparrow}{1}, 0.8, 0.7, 0.8, 1\}$ $\quad\mathrm{H}(z)=1+0.8z^{-1}+0.7z^{-2}+0.8z^{-3}+z^{-4}$





Minimum phase filters

- We call a filter a minimum phase filter, if the filter and it's inverse are both causal and stable.

$$H(z)H^{-1}(z)=1 \quad \text{if } H(z)=\frac{B(z)}{A(z)} \text{ then } H^{-1}(z)=\frac{A(z)}{B(z)}$$

- *H*(*z*) system is stable and causal if the poles (roots of *A*(*z*)) are within the unit circle, but we are free to choose the zeroes (roots of *B*(*z*)) of such system.
- The inverse of *H*(*z*) will also be stable and causal if it's poles are within the unit circle (roots of *B*(*z*))
- A filter is minimum phase if all it's poles and zeroes are inside the unit circle.

All pass filters

- A filter is said to be all pass (it passes all the frequencies with the same amplitude but may pass with different phase) if

$$\left|H\left(e^{j\omega}\right)\right|=1, \forall \omega \in [-\pi, \pi]$$

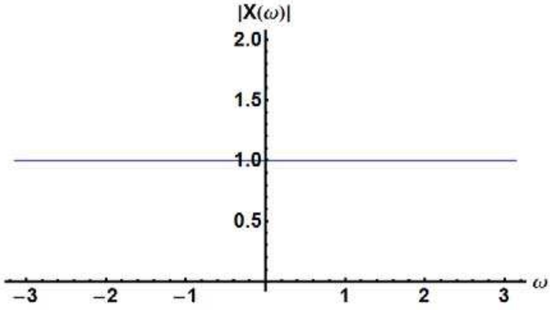
- The all pass filters have their transfer function in the form (note that real coefficients can be realized by having complex conjugate pole pairs) $H(z)=\prod_{i=1}^{N}\frac{z^{-1}-p_i^*}{1-p_i z^{-1}}$
- Example

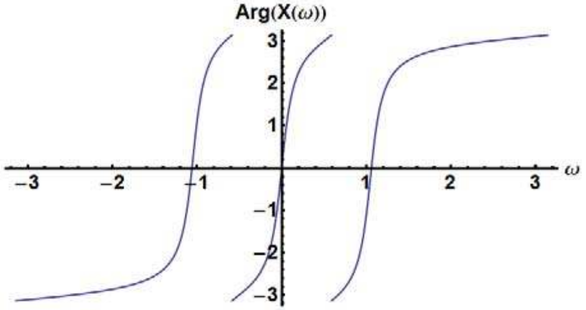
poles:

$p_1 = 0.9$

$p_2 = 0.9e^{j\pi/3}$

$p_2 = 0.9e^{-j\pi/3}$





Example 1

An LTI system is given by its system equation:

$$y(n)+y(n-2)=x(n)-2x(n-1)-0.5x(n-4)$$

a) Give the impulse response of this system, by solving it in the z-domain

b) Give the response of the system for the given stimulus by solving it in the z-domain $x(n)=2u(n)(0.5)^n$

c) Is it a BIBO stable system?

Example 1 – solution a

To give the impulse response of this system, by solving it in the z-domain we need the transfer function of the system:

$$Z\left\{y(n)+y(n-2)=x(n)-2x(n-1)-0.5x(n-4)\right\}$$

$$Y(z)+z^{-2}Y(z)=X(z)-2z^{-1}X(z)-0.5z^{-4}X(z)$$

$$Y(z)\left(1+z^{-2}\right)=X(z)\left(1-2z^{-1}-0.5z^{-4}\right)$$

$$Y(z)=X(z)\frac{1-2z^{-1}-0.5z^{-4}}{1+z^{-2}} \qquad A(z)=1+z^{-2}, \quad B(z)=1-2z^{-1}-0.5z^{-4}$$

$$\text{if } x(n)=\delta(n)\rightarrow y(n)=h(n)\overset{Z}{\rightleftharpoons}X(z)=1\rightarrow Y(z)=H(z)$$

$$H(z)=\frac{1-2z^{-1}-0.5z^{-4}}{1+z^{-2}}=\frac{1-2z^{-1}-0.5z^{-4}}{\left(1+jz^{-1}\right)\left(1-jz^{-1}\right)}, \quad \text{RoC}=\left\{z:|z|>1\right\}$$

Example 1 – solution a

We have the transfer function, we need to inverse z-transform it by either evaluating the contour integral or manipulating it to a form where we know the individual transforms of each term. We choose to do the latter by using the polynomial long division and the partial fraction expansion.

$$H(z)=\frac{1-2z^{-1}-0.5z^{-4}}{1+z^{-2}}=\frac{z^{-4}}{z^{-2}}\frac{z^{4}-2z^{-3}-0.5}{z^{2}+1}=z^{-2}\frac{z^{4}-2z^{-3}-0.5}{z^{2}+1}$$

$$\frac{z^{4}-2z^{-3}-0.5}{z^{2}+1}=z^{-2}\left(Q(z)+\frac{R(z)}{A(z)}\right)$$

Example 1 – solution a

We apply the polynomial long division:

$$\begin{array}{llllll}
 & z^4 & -2z^3 & +0z^2 & +0z-0.5:\left[z^2+1\right]=z^2 \\
-(& z^4 & +0z^3 & +1z^2) & \\
 & & -2z^3 & -1z^2 & +0z-0.5:\left[z^2+1\right]=-2z \\
-(& & -2z^3 & +0z^2 & -2z) \\
 & & & -1z^2 & +2z-0.5:\left[z^2+1\right]=-1 \\
-(& & & -1z^2 & +0z-1) \\
 & & & & +2z+0.5
\end{array}$$

$$Q(z)=z^2-2z-1 \qquad R(z)=2z+0.5$$

Example 1 – solution a

Do the partial fraction expansion to the residual part

$$Q(z) = z^2 - 2z - 1 \qquad R(z) = 2z + 0.5$$

$$H(z) = z^{M-N}\left(Q(z) + \frac{R(z)}{A(z)}\right) = z^{-2}\left(z^2 - 2z - 1 + \frac{2z+0.5}{z^2+1}\right)$$

$$\frac{R(z)}{A(z)} = \frac{2z+0.5}{z^2+1} = \frac{2z+0.5}{(z+j)(z-j)} = \frac{A}{(z+j)} + \frac{B}{(z-j)}$$

$$\frac{2z+0.5}{(z+1)(z-1)} = \frac{A(z-j)+B(z+j)}{(z+j)(z-j)} = \frac{z(A+B)+j(B-A)}{(z+j)(z-j)}$$

$$\left.\begin{aligned} 2z &= (A+B)z \\ 0.5 &= (B-A)j \end{aligned}\right\} A = 1 + 0.25j, \quad B = 1 - 0.25j$$

Example 1 – solution a

We have manipulated the transfer function to have the form:

$$H\left(z\right)=z^{M-N}\left(Q\left(z\right)+\frac{R\left(z\right)}{A\left(z\right)}\right)=z^{-2}\left(z^{2}-2z-1+\frac{1+0.25j}{\left(z+j\right)}+\frac{1-0.25j}{\left(z-j\right)}\right)$$

$$H\left(z\right)=z^{-2}\left(z^{2}-2z-1+z^{-1}\frac{(1+0.25j)z}{\left(z+j\right)}+z^{-1}\frac{(1-0.25j)z}{\left(z-j\right)}\right)$$

$$H\left(z\right)=1-2z^{-1}-z^{-2}+z^{-3}\frac{(1+0.25j)z}{\left(z+j\right)}+z^{-3}\frac{(1-0.25j)z}{\left(z-j\right)}$$

we can do the inverse z-transform easily:

$$Z^{-1}\left\{H\left(z\right)\right\}=h\left(n\right)=\delta\left(n\right)-2\delta\left(n-1\right)-\delta\left(n-2\right)+$$

$$+(1+0.25j)\left(j\right)^{n-3}u\left(n-3\right)+(1-0.25j)\left(-j\right)^{n-3}u\left(n-3\right)$$

Example 1 – solution a

Let’s compare the results with the solution got from the time domain analysis:

$$h_1(n) = h_{\text{z-transform}}(n) = \delta(n) - 2\delta(n-1) - \delta(n-2) +$$

$$+(1+0.25j)(j)^{n-3}u(n-3) + (1-0.25j)(-j)^{n-3}u(n-3)$$

$$h_2(n) = h_{\text{time dom. anal.}}(n) = +0.5\delta(n) - 0.5\delta(n-2) +$$

$$+(0.25+\mathrm{j})\,\mathrm{j}^n + (0.25-\mathrm{j})(-\mathrm{j})^n$$

it can be shown that $h_1(n) = h_2(n)$

n	0	1	2	3	4	5	6	7	8	9	10
$h_1(n)$	1	–2	–1	2	0.5	–2	–0.5	2	0.5	–2	–0.5
$h_2(n)$	1	–2	–1	2	0.5	–2	–0.5	2	0.5	–2	–0.5

Example 1 – solution b

We know the transfer function, we need to compute the z-transform of the stimulus *x*(*n*)

$$H(z)=\frac{1-2z^{-1}-0.5z^{-4}}{1+z^{-2}}$$

$$Y(z)=X(z)H(z)$$

$$X(z)=Z\left\{2u(n)(0.5)^{n}\right\}=\frac{2}{1-0.5z^{-1}},\ \text{RoC=}\left\{z:|z|>0.5\right\}$$

$$Y(z)=\frac{2}{1-0.5z^{-1}}\frac{1-2z^{-1}-0.5z^{-4}}{1+z^{-2}}=\frac{2-4z^{-1}-z^{-4}}{\left(1-0.5z^{-1}\right)\left(1-jz^{-1}\right)\left(1+jz^{-1}\right)}$$

Example 1 – solution b

We have the z-transform of the response, we need to inverse z-transform it

$$Y(z)=\frac{2}{1-0.5z^{-1}}\frac{1-2z^{-1}-0.5z^{-4}}{1+z^{-2}}=\frac{2-4z^{-1}-z^{-4}}{\left(1-0.5z^{-1}\right)\left(1-jz^{-1}\right)\left(1+jz^{-1}\right)}=$$

$$\frac{2z^{-4}}{z^{-3}}\frac{z^4-2z^3-0.5}{(z-0.5)(z-j)(z+j)}=2z^{-1}\left(Q(z)+\frac{R(z)}{A(z)}\right)$$

after polynomial long division:

$$Q(z)=z-1.5,\quad R(z)=-1.75z^2+2z-1.25$$

after partial fraction

$$\frac{R(z)}{A(z)}=-\frac{0.55}{z-0.5}-\frac{0.6+0.7j}{z-j}-\frac{0.6-0.7j}{z+j}$$

Example 1 – solution b

We have manipulated the response to have the form:

$$Y\left(z\right)=z^{k}\left(Q\left(z\right)+\frac{R\left(z\right)}{A\left(z\right)}\right)=2z^{-1}\left(z-1.5-\frac{0.55}{\left(z-0.5\right)}-\frac{0.6-0.7j}{\left(z+j\right)}-\frac{0.6+0.7j}{\left(z-j\right)}\right)$$

$$Y\left(z\right)=2z^{-1}\left(z-1.5-z^{-1}\frac{0.55z}{\left(z-0.5\right)}-z^{-1}\frac{\left(0.6-0.7j\right)z}{\left(z+j\right)}-z^{-1}\frac{\left(0.6+0.7j\right)z}{\left(z-j\right)}\right)$$

$$Y\left(z\right)=2-3z^{-1}-z^{-2}\frac{1.1z}{\left(z-0.5\right)}-z^{-2}\frac{\left(1.2-1.4j\right)z}{\left(z+j\right)}-z^{-2}\frac{\left(1.2+1.4j\right)z}{\left(z-j\right)}$$

we can do the inverse z-transform easily:

$$Z^{-1}\left\{Y\left(z\right)\right\}=y\left(n\right)=2\delta\left(n\right)-3\delta\left(n-1\right)-$$

$$-1.1\cdot\left(0.5\right)^{n-2}u\left(n-2\right)-\left(1.2-1.4j\right)\left(-j\right)^{n-2}u\left(n-2\right)-\left(1.2+1.4j\right)j^{n-2}u\left(n-2\right)$$

Example 1 – solution b

Let’s compare the results with the solution got from the time domain analysis:

$$y_1(n) = y_{\text{z-transform}}(n) = 2\delta(n) - 3\delta(n-1) - 1.1\cdot(0.5)^{n-2}\,u(n-2) -$$

$$-(1.2-1.4j)(-j)^{n-2}\,u(n-2) - (1.2+1.4j)\,j^{n-2}u(n-2)$$

$$y_2(n) = y_{\text{time dom. anal.}}(n) = +4\delta(n) + 2\delta(n-1) +$$

$$+(1.2+1.4\text{j})\,\text{j}^n + (1.2-1.4\text{j})(-\text{j})^n - 4.4\cdot u(n)(0.5)^n$$

it can be shown that $y_1(n) = y_2(n)$

n	0	1	2	3	4	5	6	7
$y_1(n)$	2	-3	$-7/2$	$9/4$	$17/8$	$-47/16$	$-79/32$	$177/64$
$y_2(n)$	2	-3	$-7/2$	$9/4$	$17/8$	$-47/16$	$-79/32$	$177/64$

Example 1 – solution c

c) For the system to be BIBO stable the necessary condition is to have the system transfer function’s poles within the unit circle.

$$H\left(z\right)=\frac{1-2z^{-1}-0.5z^{-4}}{1+z^{-2}}=\frac{1-2z^{-1}-0.5z^{-4}}{\left(1+jz^{-1}\right)\left(1-jz^{-1}\right)}$$

poles: $p_1 = j, \quad p_2 = -j$

The poles are exactly at the unit circle, so the system is **not** BIBO stable.

Summary of system specific functions and properties

Time domain	excitation $x(n)$	response $y(n)$	Impulse response $h(n):=h(n)*\delta(n)$	System: $y(n)=h(n)*x(n)$
Transformed, z-domain	$X(z)$	$Y(z)$	Transfer function $H(z)$ $H(z)=\frac{Y(z)}{X(z)}$	System: $Y(z)=H(z)\cdot X(z)$ Pole-zero plot: $H(z)=\dfrac{B(z)}{A(z)}\begin{matrix}\leftarrow\text{zeros}\\ \leftarrow\text{poles}\end{matrix}$
Frequency (Fourier) domain	Spectrum of the excitation: $X(\omega),X(f)$ $X\in\mathbb{C}$	Spectrum of the response: $Y(\omega),Y(f)$ $Y\in\mathbb{C}$	Spectrum: $H(\omega),H(f)$ Amplitude, phase char: $\lvert H(\omega)\rvert,\arg(H(\omega))$ $H\in\mathbb{C}$	Bode plot: freq. vs. ampl or freq. vs. phase plot Nyquist plot: parametric plot of $H(\omega)$ in the complex plane or equivalently in polar coords: phase vs. amp.

Summary

- Introduction and illustration of the z-transformation.
- Properties of the z-transform.
- Elementary signals transforms.
- Use of the z-transform in the analysis of the LTI systems.
- Filter realization forms based on the z-transform properties.
- DFT and z-transform connection.
- Examples.