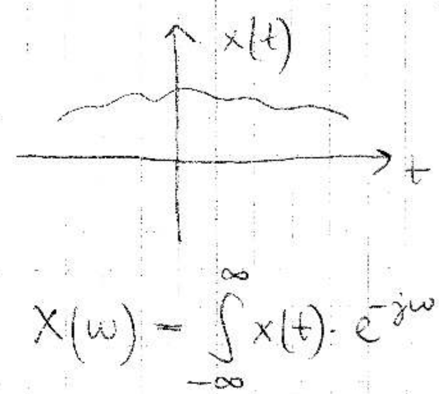
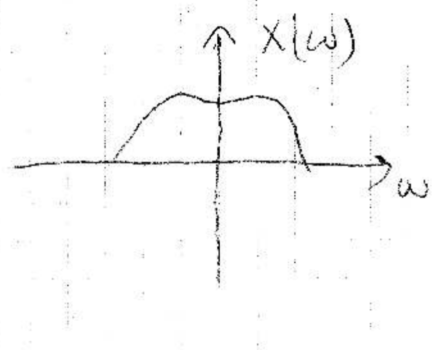
Spektrális analízis



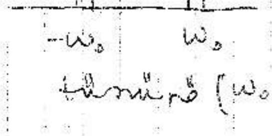


$\longrightarrow$ sávszélesség, energia $\longrightarrow$ mérnöki tervezés

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

Old-timer approach

$$x(t) \longrightarrow \underset{}{\overset{H(\omega)}{\boxed{h(t)}}} \longrightarrow y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$



$$Y(\omega) = H(\omega) X(\omega) \approx X(\omega_0)$$

sávszűrő (ω_0 állítható)

Ehelyett

$$x(t) \longrightarrow \boxed{A/D} \xrightarrow{x(n)} \boxed{DSP} \longrightarrow X(\omega)$$

?

működik-e ez a módszer?

$$X_D(\omega) := \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega} = X(z)\Big|_{z=e^{j\omega}}$$ mintavételezett jel ($x(n)$) spektruma

$$X_D(\omega) = \sum_{l} X\left(\omega + l\frac{2\pi}{T}\right)$$

Ha $f_s \geq 2B \Rightarrow X_D(\omega) = X(\omega)$ $\quad -\frac{1}{2T} < f < \frac{1}{2T}$

Következmény: ha $X_D(\omega)$-t megismerjük, akkor $X(\omega)$-t is ismerjük, fordítva is igaz.

ω (körfrekv.) folytonos változó $\rightarrow$ $k\frac{2\pi}{N}$ (diszkrét változó)

$$X_D(\omega) \Longrightarrow X\left(k\frac{2\pi}{N}\right) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-jnk\frac{2\pi}{N}} = (\infty \text{ összeg})$$

$$= \dots + \sum_{n=-N}^{-1} x(n)\, e^{-jnk\frac{2\pi}{N}} + \sum_{n=0}^{N-1} x(n) \cdot e^{-jnk\frac{2\pi}{N}} + \sum_{n=N}^{2N-1} x(n)\, e^{-jnk\frac{2\pi}{N}} + \dots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{(l+1)N-1} x(n) e^{-jnk\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x_p(n) e^{-jnk\frac{2\pi}{N}}$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

→ periodikus fv (N) → Fourier-sorba fejthető

→ $N > \text{support}\{x(n)\} \rightarrow x_p(n) = x(n)$ $n=0, \dots N-1$

$$x_p(n) = \sum_k c_k \cdot e^{-jnk\frac{2\pi}{N}} \qquad c_k = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-jnk\frac{2\pi}{N}} = \frac{1}{N} X\left(k\frac{2\pi}{N}\right)$$

$(x_p(n))$ mert csak erről az int. tartományról van szó

$$X(\omega) = \sum_{n=0}^{N-1} x(n) \cdot e^{-jn\omega} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \frac{1}{N} X\left(k\frac{2\pi}{N}\right) e^{jnk\frac{2\pi}{N}} e^{-jn\omega} =$$

mert véges tartójú a jel

$$= \sum_{k=0}^{N-1} X\left(k\frac{2\pi}{N}\right) \frac{1}{N} \sum_{n=0}^{N-1} e^{-jn(\omega - k\frac{2\pi}{N})}$$

$$\boxed{\begin{aligned} X(\omega) &= \sum_{k=0}^{N-1} \underbrace{X\left(k\frac{2\pi}{N}\right)}_{x(n)} \underbrace{P\left(\omega - k\frac{2\pi}{N}\right)}_{\text{simítófv}} \\ P(\omega) &= \frac{1}{N} \sum_{n=0}^{N-1} e^{-jn\omega} = \frac{1}{N} \frac{1-e^{-jN\omega}}{1-e^{-j\omega}} \\ X\left(k\frac{2\pi}{N}\right) &= \sum_{n=0}^{N-1} x(n) \cdot e^{-jnk\frac{2\pi}{N}} \end{aligned}}$$

$X\left(k\frac{2\pi}{N}\right) = X(k)$ → jelölés!

$$\rightarrow X(k) = \sum_{n=0}^{N-1} x(n) e^{-jnk\frac{2\pi}{N}} \qquad k = 0, \dots N-1 \rightarrow \overline{X} \quad \text{DFT}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{jnk\frac{2\pi}{N}} \qquad n = 0, \dots N-1$$

$$W(n) = e^{-j\frac{2\pi}{N}}$$

$$X(k) = \sum_{n=0}^{N} x(n) W(n)^{kn} \qquad k = 0, \dots N-1$$

$$x(n) = \sum_{k=0}^{N-1} X(k) W(n)^{-kn} \qquad n = 0, \dots N-1$$

$$\left.\begin{aligned} \overline{X} &= \overline{\overline{W}}\,\overline{x} \rightarrow x = \overline{\overline{W}}^{-1} \overline{X} \\ \overline{x} &= \frac{1}{N} \overline{\overline{W}}^{*T} \overline{X} \end{aligned}\right\} \quad \overline{\overline{W}}^{-1} = \frac{1}{N} \overline{\overline{W}}^{*T} \rightarrow N\overline{\overline{I}} = \overline{\overline{W}}^{*T} \overline{\overline{W}}$$

(ortogonális / unitér mátrix)

Spektrálanalízis

$$x(t) \rightarrow \boxed{A/D} \xrightarrow[n=0,\dots N-1]{x(n)} \boxed{\overline{\overline{W}}} \xrightarrow[k=0,\dots N-1]{X(k)} \boxed{} \rightarrow X(\omega)$$

$$\sum_{k=0}^{N-1} X(k)\, P\left(\omega - k\frac{2\pi}{N}\right)$$

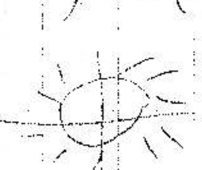
Ez ~~egy~~ mind real-time művelet!!

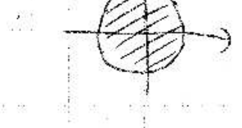
Gyakorlat: IV. 17.

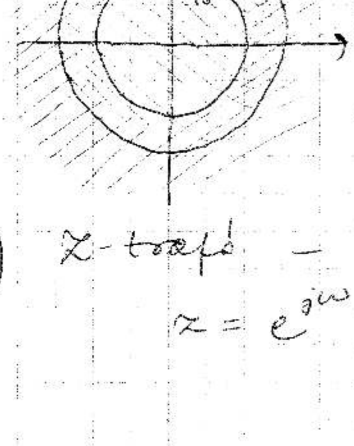
1.) Rac. törtfv: pólusok az egységkörön belül → stabil

2.) $H(z) = z^1 + 1 \cdot z^0$ antikauzális?

Igen, mert $h(n) = \delta(n+1) + \delta(n)$

kauzális $H(z) = \frac{z}{z-a}$ 

akauzális 

3.) 

4.) Z-trafó — Fourier-trafó

$z = e^{j\omega}$ ω az egys.körön mozog
ha az egységkör nincs a ROC-ban ⇒ a z transzformálható fv. nem Fourier-transzformálható!

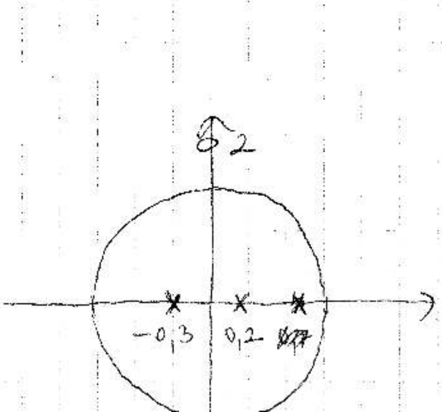
1.) $y(n) + 0,1\, y(n-1) - 0,06\, y(n-2) = x(n) + 4\, x(n-2)$

$x(n) = u(n) \cdot \sin(0,5n)$

$y(-2) = y(-1) = 0$

$$H(z) = \frac{1+4z^{-2}}{1+0,1z^{-1}+0,06z^{-2}} = \frac{z^2+4}{z^2+0,1z-0,06}$$

zérusok: $z_{1,2}$ $+2i$, $-2i$



pólusok: p $-0,3$

$$H(z) \xRightarrow{\mathcal{Z}^{-1}\{\}} h(n)$$

$$H(z) = \frac{z^2+4}{(z+0{,}3)(z-0{,}2)} = Q(z) + \frac{R(z)}{A(z)} = 1 + \frac{-0{,}1z+4{,}06}{z^2+0{,}1z-0{,}06} =$$

$$\begin{cases} z^2 + 0z + 4 : z^2+0{,}1z-0{,}06 = 1 \\ \underline{z^2+0{,}1z-0{,}06} \\ 0 - 0{,}1z + 4{,}06 \end{cases}$$

$$= 1 + \frac{A_1}{z+0{,}3} + \frac{A_2}{z-0{,}2} = 1 + \frac{-8{,}18}{z+0{,}3} + \frac{8{,}08}{z-0{,}2} =$$

$$A_1 = \left.\frac{-0{,}1z+4{,}06}{z-0{,}2}\right|_{z=-0{,}3} = -8{,}18$$

$$A_2 = \left.\frac{-0{,}1z+4{,}06}{z+0{,}3}\right|_{z=0{,}2} = 8{,}08$$

$$\frac{z}{z-a} \Rightarrow a^n \cdot u(n)$$

$$= 1 - 8{,}18z^{-1}\cdot\frac{z}{z-(-0{,}3)} + 8{,}08z^{-1}\frac{z}{z-0{,}2}$$

$$h(n) = \delta(n) - \varepsilon(n-1)(-0{,}3)^{n-1}\cdot 8{,}18 + 8{,}08\cdot(0{,}2)^{n-1}\cdot\varepsilon(n-1)$$

$$\mathcal{Z}\{u(n)\cdot\sin(\omega_0 n)\} = \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cdot\cos(\omega_0)+z^{-2}} = \frac{z\sin(\omega_0)}{z^2-2z\cos(\omega_0)+1}$$

$$\cos(\omega_0 n) \qquad \frac{z\cdot\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$$

$$y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\{H(z)\cdot X(z)\} = \mathcal{Z}^{-1}\left\{\frac{B(z)}{A(z)}\cdot X(z)\right\}$$

$$Y(z) = \frac{z^3\cdot\sin(0{,}5) + 4z\sin(0{,}5)}{(z+0{,}3)(z-0{,}2)\underbrace{(z^2-2z\cos(0{,}5)+1)}_{\text{irreducibilis polinom } n\text{-ed fokú}}} = \frac{A_1}{z+0{,}3} + \frac{A_2}{z-0{,}2} + \frac{A_3z+A_4}{z^2-2z\cos(0{,}5)+1}$$

→ n-1-ed fokú számlálót veszünk

$$A_1 = \left.\frac{z^3\sin(0{,}5)+4z\cdot\sin(0{,}5)}{(z-0{,}2)(z^2-2z\cos(0{,}5)+1)}\right|_{z=-0{,}3} = 0{,}728$$

$$A_2 = \left.\frac{z^3\sin(0{,}5)+4z\sin(0{,}5)}{(z+0{,}3)(z^2-2z\cos(0{,}5)+1)}\right|_{z=0{,}2} = 1{,}124$$

A_3: Lagrange-féle parciális törtekre bontással

$$A_1(z-0{,}2)(z^2-2z\cos(0{,}5)+1)+A_2(z+0{,}3)(z^2-2z\cos 0{,}5+1)+(A_3z+A_4)(z-0{,}2)(z+0{,}3)$$

ismerjük már: $0{,}191795 - 0{,}06\cdot A_4 + \quad \to A_4 = \frac{0{,}191795}{0{,}06} = +3{,}196$

$$+z^3(1{,}8523+A_3)+$$
$$+z^2(-3{,}0593+0{,}1A_3+A_4)$$
$$+z(1{,}5156-0{,}06A_3+0{,}1A_4) = z^3\sin 0{,}5+4z\sin 0{,}5$$

$$A_3 = -1{,}372$$

$$Y(z) = \frac{0{,}728}{z+0{,}3}+\frac{+1{,}124}{z-0{,}2}+\underbrace{\frac{-1{,}372z+3{,}196}{z^2-2z\cos 0{,}5+1}}_{*}$$

$\Downarrow z^{-1}$

$$y(n) = \underbrace{0{,}728\cdot(-0{,}3)^{n-1}}_{\cdot\varepsilon(n-1)} + \underbrace{1{,}124(0{,}2)^{n-1}}_{\cdot\varepsilon(n-1)} + \frac{-1372}{\sin 0.5}\cdot\sin(0.5n)\varepsilon(n) + \frac{3{,}196}{\sin 0.5}\sin(0.5(n-1))\varepsilon(n-1)$$

$$* = \frac{-1{,}372z}{z^2-2z\cos 0{,}5+1}+\frac{3{,}196}{z^2-2z\cos 0{,}5+1} = \frac{-1{,}372}{\sin(0.5)}\cdot\frac{(\sin 0.5)z}{-\,''\,-}+$$

$$+\frac{3{,}196}{\sin(0{,}5)}z^{-1}\cdot\frac{\sin(0.5)}{\sin(0.5)}$$

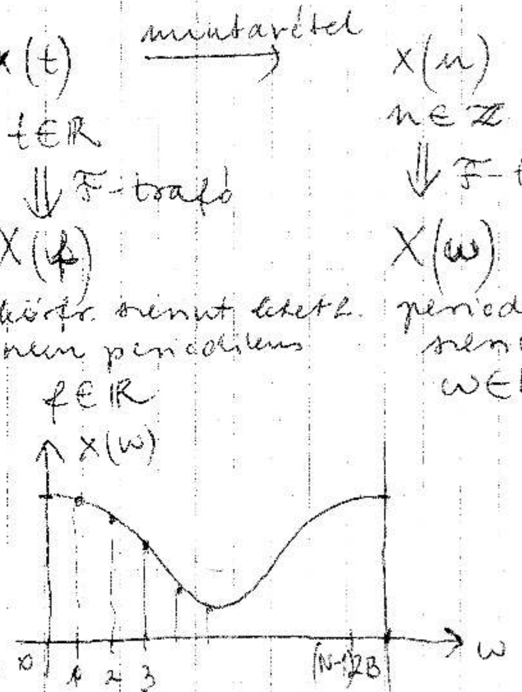
$\cancel{\frac{-1{,}372z}{\sin 0{,}5}\sin 0.5\,\varepsilon(n)} +$

Diszkrét Fourier-transzformáció

$$\begin{array}{ccccc} x(t) & \xrightarrow{\text{mintavétel}} & x(n) & & \\ t\in\mathbb{R} & & n\in\mathbb{Z} & \xrightarrow{\text{DFT}} & \\ \Downarrow \mathcal{F}\text{-trafó} & & \Downarrow \mathcal{F}\text{-trafó (DTFT)} & & \\ X(f) & & X(\omega) & \xrightarrow{\text{mintavétel}} & X\left(k\frac{2\pi}{N}\right) \end{array}$$

körfr. szerint létezik, nem periodikus, $f\in\mathbb{R}$

periodikus ω szerint, $\omega\in\mathbb{R}$



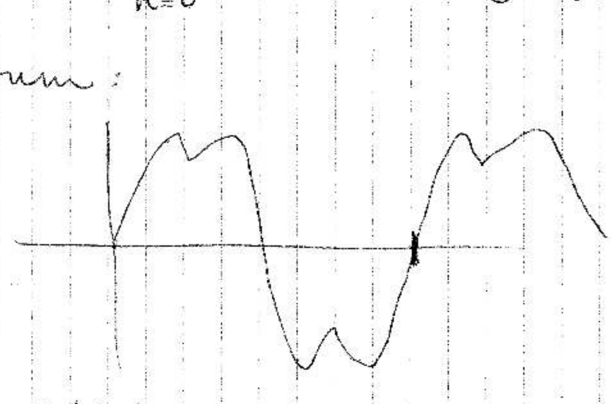
) $x(n) = \varepsilon(n) a^n \qquad 0 < a < 1$

$\omega_k = \frac{2\pi}{N} k \qquad k = 0 \dots N-1$

$X(\omega) = \sum_{n=-\infty}^{\infty} \varepsilon(n) a^n \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n} = \frac{e^{j\omega}}{e^{j\omega}-a} = \frac{1}{1-a e^{-j\omega}}$

Mintavett spektrum:

$x_p(n) = \sum_{l=0}^{N-1} x(n-lN)$



$x_p(n) = x(n) \qquad 0 < n < N-1$

$x_p(n) = \sum_{l=-\infty}^{\infty} x(n+lN) = \sum_{l=\infty}^{0} a^{n-lN} = a^n \sum_{l=0}^{\infty} a^{lN} = \frac{1}{1-a^N} \cdot a^n = \frac{a^n}{1-a^N}$

$X(\omega_k) = \frac{1}{1-a \cdot e^{-j\frac{2\pi}{N}k}}$

) $x(n) = \begin{cases} 1 & 0 < n < L-1 \\ 0 & \text{máshol} \end{cases}$

N pontos DFT

$X(\omega) = \sum_{n=0}^{L-1} x(n) \cdot e^{-j\omega n} = \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega \frac{L}{2})}{\sin(\omega/2)} \cdot e^{-j\omega(\frac{L-1}{2})}$

$X(k) = \frac{\sin\left(\frac{2\pi}{N} k \frac{L}{2}\right)}{\sin\left(\frac{\pi}{N} k\right)} \cdot e^{-j\frac{2\pi}{N}k\left(\frac{L-1}{2}\right)}$

$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad W_N = e^{-j\frac{2\pi}{N}}$

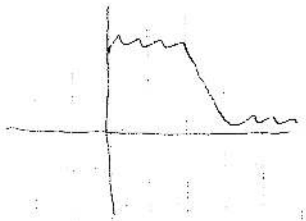
$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-nk}$

$\Downarrow$ mátrix-vektor szorzással felírva:

$\overline{X}_N = \underline{x} \cdot \underline{\underline{W}}_N$

$\underline{x} = \frac{1}{N} \boxed{\overline{W}_N^*} \overline{X}_N$

Szűrőtervezés: Adott

 $\Rightarrow \delta, \omega_c, \omega_d, \varepsilon$

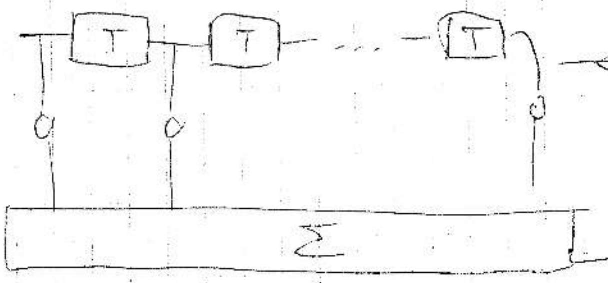
$\Downarrow$

$w(n)$ választása úgy, hogy

$\downarrow$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \quad (\text{minn } N) \longleftarrow$$

$$h(n) = w(n) \cdot h_d(n)$$



Ha nem aluláteresztő szűrőt akarunk $\Rightarrow$ az első lépés egy transzformáció (az $x(t)$-t pl. megfordítjuk)

IV. 24.

Gyakorlat

$x(t) \xrightarrow{\text{minta}} x(n)$ — DFT → $X(k)$

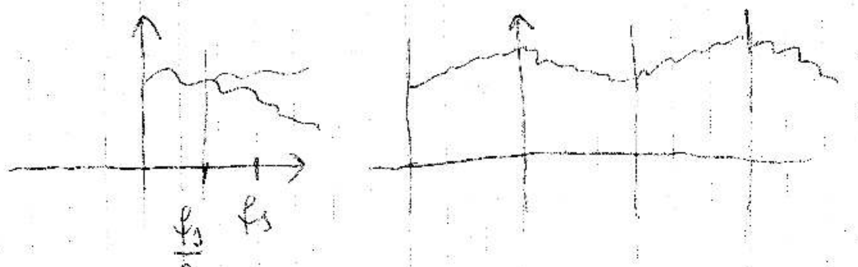
$\Downarrow$ F-tr. $\quad \Downarrow$ DTFT: diszkr. idejű Fourier-transzf.

$X(f) \quad X_D(f) \xrightarrow{\text{minta}} X(k)$

$X(\omega) \quad X_D(\omega) \longleftarrow$ szorzás sinc függvénnyel

$\omega = 2\pi f$

$f, \omega \in \mathbb{R} \qquad f, \omega \in \mathbb{R}$



$x(n) \longrightarrow \overline{x} \in \mathbb{R}^n \qquad n = 0 \dots N-1$

$\overline{X} = \overline{\overline{W}}_N \cdot \overline{x}$

$\left(\overline{\overline{W}}_N\right)_{n,k} = W_N^{kn} \Leftrightarrow e^{-j\frac{2\pi}{N}} = W_N$

$\overline{x} = [0, 1, 2, 3]$

4 pontos DFT ?

$$\overline{\overline{W}}_4 = \begin{array}{c c} & n \\ m & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \qquad W_{N=4}^{n \cdot m} = \left(e^{-j\frac{2\pi}{4}}\right)^{n \cdot m}$$

$$\begin{pmatrix} \\ \\ \end{pmatrix} \to \begin{pmatrix} \\ \\ \end{pmatrix}$$

$N \qquad N = L \cdot M$

$2 \cdot \boxed{4}$

$\text{rem}(N/2) = 0 \quad 2 \cdot \boxed{2} \boxed{2}$

$2^{\nu} = N$

$N = 8 = 2^3$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} = \sum_{n:\text{pós}} x(n) \cdot W_N^{kn} + \sum_{n:\text{ptlan}} x(n) W_N^{kn} =$$

$n = 2m \qquad n = 2m+1$

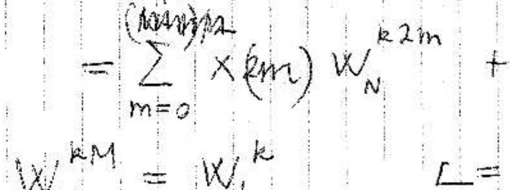
$m = \frac{n}{2} \qquad m = \frac{n-1}{2}$

$$= \sum_{m=0}^{N/2-1} x(2m) W_N^{k2m} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{k(2m+1)}$$

$W_N^{kM} = W_L^k \qquad L = \frac{N}{M}$

$\mapsto$ pl.: $W_8^k = W_4^{2k}$

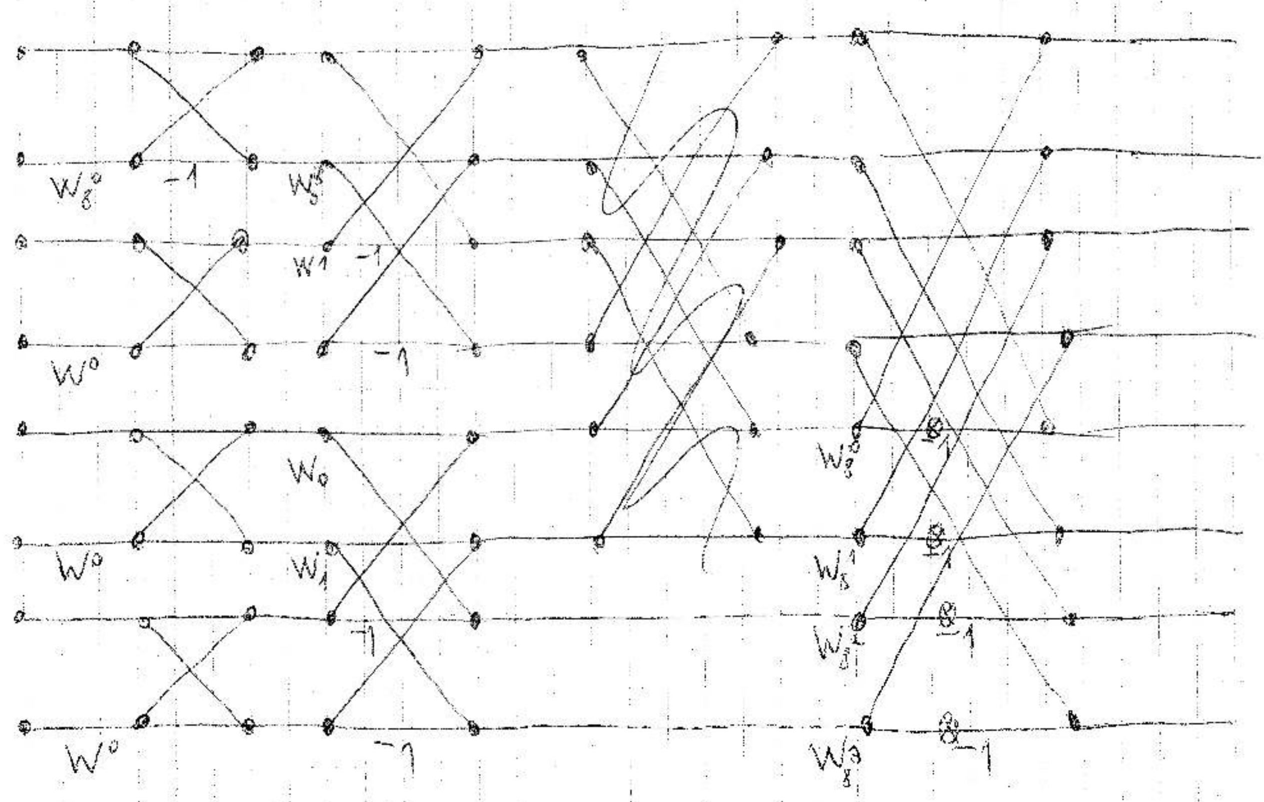
2 pontos DFT



$a + W_N^k \cdot b$

$a - W_N^k \cdot b$

$8 = 2^3$



1 2 2 1 | 0 0 0 0   zero padding - 0-val való kiegészítés

↑ ↑ ↑ ↑

1200   2100

10   20   20   10

(1.) $x(n) = u(n) \cdot a^n \quad a = 0{,}8$

$\omega_k = \frac{2\pi}{N} k \qquad N = 50$

$k = 0 \dots N-1$

(2.)

$K = 100$
$L = 10$
$N = 50$

```
K = 100; L = 10; N = 50
X = zeros(1, 100)
X(1:L) = 1
X = fft(X)
mask = zeros(1, K)
mask(1 : K/N : K) = 1
Xk = mask X
```