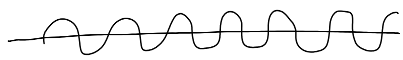
Hullámegyenlet (wave-eq.)

2015. december 14., hétfő 16:18

$u(x,t)$: $\left.\begin{array}{l} t \text{ időben} \\ x \text{ helyen} \end{array}\right\}$ $u''_{tt} = c^2 u''_{xx}$ (WE)
wave-equation

Végtelen húr rezgése: $t \geq 0$, $x \in \mathbb{R}$ 

Nincs peremfeltétel, csak kezdeti feltétel

$u(x,0) = f(x)$

$u'_t(x,0) = g(x)$

D'Alambert-féle megoldás

$(x,t) \mapsto (\xi, \eta)$ (koordináta-transzformáció)

$\xi = x + ct \Rightarrow x = \frac{\xi + \eta}{2}$

$\eta = x - ct \Rightarrow t = \frac{\xi - \eta}{2c}$

$$U(\xi,\eta) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$$

$$U'_\xi(\xi,\eta) = u'_x \cdot \frac{1}{2} + u'_t \cdot \frac{1}{2c}$$

$$\frac{\partial}{\partial\eta} U'_\xi(\xi,\eta) = u''_{xx}\frac{1}{4} + \cancel{u''_{xt}\frac{1}{2}\left(-\frac{1}{2c}\right)} + \cancel{u''_{tx}\frac{1}{2c}\cdot\frac{1}{2}} + u''_{tt}\left(-\frac{1}{4c^2}\right) = \frac{1}{4c^2}\left(c^2 u''_{xx} - u''_{tt}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial\eta} U'_\xi(\xi,\eta) = 0$$

$$U'_\xi(\xi,\cancel{\eta}) = F_0(\xi)$$

$$U(\xi,\eta) = \int F_0(\xi)\,d\xi = F(\xi) + G(\eta)$$

$\Rightarrow u(x,t) = U(x+ct, x-ct) = F(x+ct) + G(x-ct)$ -> visszaváltoztattuk a koordinátákat

Megjegyzés:

Diff. operátorral: $L[u] = 0$

$$L = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$$

$L = L_1 \cdot L_2 \xrightarrow[\text{felcserélhető}]{\text{szorzások sorrendje}}$ $L[u] = L_1(L_2[u]) = L_2(L_1[u]) = 0$

$$L[u] = 0 \Leftarrow \begin{cases} L_1[v] = 0 \\ L_2[w] = 0 \end{cases}$$

$L_1[v] = v'_t - c v'_x = 0$: $v(x,t) = a(x+ct)$, $a: \mathbb{R} \to \mathbb{R}$
$L_2[w] = w'_t + c w'_x = 0$: $w(x,t) = b(x-ct)$, $b: \mathbb{R} \to \mathbb{R}$
} két transzport egyenlet

Kezdeti feltételek felhasználása

$u = u_0 + u_1$ (u_0 és u_1 is teljesíti a WE-t)

① $u_0(x,0) = f(x)$ | ② $u_1(x,0) = 0$

$u = u_0 + u_1$ (u_0 és u_1 is teljesíti a WE-t)

① $u_0(x,0) = f(x)$	② $u_1(x,0) = 0$
$u'_{0t}(x,0) = 0$	$u'_{1t}(x,0) = g(x)$

$\downarrow$

① $u_0(x,t) = F(x+ct) + G(x-ct)$

$F(x) + G(x) = f(x) \quad \rightarrow \quad F(x) = \frac{f(x)}{2}$

$F'(x)\cdot c - c\,G'(x) = 0$

$F'(x) = G'(x) \Rightarrow F(x) = G(x)$

$$\hookrightarrow \boxed{u_0(x,t) = \frac{f(x+ct) + f(x-ct)}{2}}$$

② $u_1(x,t) = F_1(x+ct) + G_1(x-ct)$

$F_1(x) + G_1(x) = 0 \Rightarrow G_1(x) = -F_1(x)$

$u'_{1t}(x,t) = F_1'(x) - ((-c)F_1'(x)) = g(x)$

$2c\,F_1'(x) = g(x)$

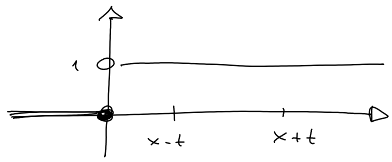
$F_1(x) = \frac{1}{2c}\int_0^x g(s)\,ds$

$$\hookrightarrow \boxed{u_1(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

D'Alambert-féle mo.: $\boxed{u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$

Példák:

1) $f(x) = \begin{cases} 1 & : x > 0 \\ 0 & : x \le 0 \end{cases}$



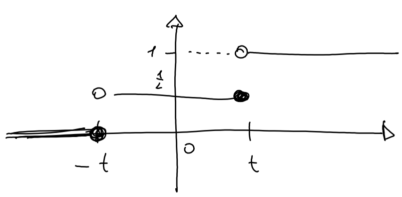
$g(x) = 0$

$c = 1$

$\Rightarrow u''_{tt} = u''_{xx}$

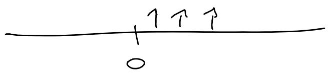
$u(x,t) = \frac{f(x+t) + f(x-t)}{2}$

Rögzített t-re: $\begin{cases} 0 & : x \le -t \\ \frac{1}{2} & : -t < x \le t \\ 1 & : x > t \end{cases}$



2) $f(x) = 0$

$$g(x) = \begin{cases} 1 & : x > 0 \\ 0 & : x \le 0 \end{cases}$$



$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} g(s)\,ds = \begin{cases} 0 & : x \le -t \\ \frac{x+t}{2} & : -t < x \le t \\ t & : x > t \end{cases}$$

Rögzített t-re: