

VARIÁCIÓSZÁMITÁS

$$\frac{\delta F}{\delta u} - \frac{\delta}{\delta x} \frac{\delta F}{\delta u'} = 0 \quad \rightarrow \text{Euler-egyenlet} \quad L(u) = 0$$

ha nem függ x -től közvetlenül: $E(x) = F - u' \cdot \frac{\delta F}{\delta u'} = c$

másképp fel lehet írni $L(u)$ -t és $E(x)$ -et, és a kiegészítő megoldani.

feltétel: $\int_I G \, dx = 0$ a feltétel

$$\tilde{F} = F - \lambda \cdot G \quad \tilde{F}\text{-re csinálunk a Lagrange megoldást}$$

λ -t megkapom, ha viselkedésként a G -t integrálom $\rightarrow$ integrálom

PARCIA LIS DIFFERENCIÁLEGYENLETEK

transport: $u_t + bu'_x = 0 \quad \rightarrow b\text{-t kioldom, felírom}$

$$u(x, 0) = g(x) \quad \swarrow \text{homogén}$$

ígyem: $u(x, t) = g(x - bt)$ ✓

inhomogén: $u_t + b \cdot u'_x = f(x, t)$

$$u(x, 0) = g(x)$$

m.o.: $u(x, t) = v(x, t) + w(x, t)$

$$v_t + b \cdot v'_x = 0$$

$$v(x, 0) = g(x)$$

↓

$$v(x, t) = g(x - bt)$$

$$w_t + b \cdot w'_x = f(x, t)$$

$$w(x, 0) = 0$$

$$w(x, t) = \int_{-t}^0 f(x + bs, t + s) \, ds$$

végül pedig ellenőrzöm

Laplace: $u''_{xx} + u''_{yy} = 0 \quad \Delta u = 0$

megyeket szorzat peremfeltételek kenne

$u(x,y) = X(x) \cdot Y(y)$ alakban keressük

$\Downarrow$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda_0 \quad \begin{cases} \lambda_0 = \alpha^2 > 0 \\ \lambda_0 = -\alpha^2 < 0 \end{cases}$$

kell hozzá Fourier: $f(x) = \sum_{k=0}^{\infty} a_k \cdot \cos(k\pi x) + b_k \cdot \sin(k\pi x)$

$$a_k = \int_{-p}^p f(x) \cdot \cos(k\pi x) dx$$

$$b_k = \frac{2}{p} \int_p^p f(x) \cdot \sin(k\pi x) dx$$

$\sum u_k(x,t)$ szorzat kenne

hővezetés: $u'_t = u''_{xx} \quad (0,1) \times (0,\infty)$

$u(x,0) = f(x)$ $\uparrow$ véges mértékben

$u(0,t) = 0$

$u(1,t) = 0$ $\leftarrow$ homogén feltételeknek kell kenne

$u(x,t) = X(x) \cdot T(t)$ alakban + megpróbáljuk kielégíteni a feltételeket

$u'_t = u''_{xx} \quad x \in \mathbb{R}, t > 0$

kezdeti hőmértéklet: $u(x,0) = f(x)$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_{-\infty}^{\infty} f(y) \cdot e^{-\frac{(y-x)^2}{4t}} dy$$

~~$f(x) = \delta(x - \sqrt{2t})$~~ normál eloszlás sűrűség függvényéhez hasonlít

9) $u_t' = u_{xx}'' \quad (0,1) \times (-\infty, 0)$

$$u(x, 0) = x^2 - x \quad 0 < x < 1$$

$$u(0, t) = 0 \quad t < 0$$

$$u(1, t) = 0 \quad t < 0$$

$$s = -t$$

$$-u_s' = u_{xx}''$$

$$u(x, 0) = x^2 - x \quad 0 < x < 1$$

$$u(0, s) = 0 \quad s > 0$$

$$u(1, s) = 0 \quad s > 0$$

$$u(x, s) = X(x) \cdot S(s) \quad -X \cdot S' = X'' \cdot S$$

$$-\frac{S'}{S} = \frac{X''}{X} = \lambda_0$$

1) $\lambda_0 = \alpha^2 > 0$

$$-\frac{S'}{S} = \alpha^2 \quad -S' - \alpha^2 S = 0$$

$$-\lambda - \alpha^2 = 0$$

$$\lambda = -\alpha^2$$

$$S(s) = C_1 e^{-\alpha^2 s}$$

$$\frac{X''}{X} = \alpha^2$$

$$X'' - \alpha^2 X = 0$$

$$\lambda^2 = \alpha^2$$

$$\lambda = \pm \alpha$$

$$X(x) = C_2 e^{\alpha x} + C_3 e^{-\alpha x}$$

$$X(0) = C_2 + C_3 = 0 \rightarrow C_3 = -C_2$$

$$X(1) = C_2 e^{\alpha} + C_3 e^{-\alpha} = 0$$

$$C_2 (e^{\alpha} - e^{-\alpha}) = 0 \quad \begin{cases} C_2 = 0 \\ \alpha = 0 \end{cases}$$

2) $\lambda_0 = -\alpha^2 < 0$

$$\lambda = \alpha^2 \rightarrow S(s) = C_1 e^{\alpha^2 s}$$

$$\lambda = \pm \alpha i \rightarrow X(x) = A \cos(\alpha x) + B \sin(\alpha x)$$

$$X(0) = A = 0$$

$$X(1) = B \sin(\alpha) = 0 \quad \begin{cases} B = 0 \\ \alpha = k\pi \quad k \in \mathbb{Z} \end{cases}$$

$$u(x, t) = \sum_{k=0}^{\infty} A_k e^{-(k\pi)^2 t} \sin(k\pi x) \rightarrow u(x, t) = \sum_{k=0}^{\infty} A_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

Ans. 1.1.1

11.

$$u_t = u_{xx}$$

$$u(x, 0) = \sin(\pi x) \quad 0 < x < 1$$

$$u(0, t) = 0 \quad t > 0$$

$$u_x(1, t) = 0 \quad t > 0$$

11. $u_t(x, t) = u_{xx} \quad x \in \mathbb{R} \quad t > 0$

$$u(x, 0) = f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & |x| > 1 \end{cases}$$

$$u(x, t) = X(x) \cdot T(t)$$

Ans.

$$T(t) = C \cdot e^{\lambda t}$$

$$X(x) = C_1 e^{sx} + C_2 e^{-sx}$$

$$u(x, t) = A \cdot e^{sx + s^2 t} + B \cdot e^{-sx + s^2 t}$$

$$u(x, 0) = A \cdot e^{sx} + B \cdot e^{-sx} \quad \lambda = s^2$$

11) $\lambda = -s^2 \quad X(x) = A \cdot \cos(sx) + B \cdot \sin(sx)$

$$u(x, t) = (A \cdot \cos(sx) + B \cdot \sin(sx)) e^{-s^2 t}$$

$$u(x, t) = \sum_{k=0}^{\infty} (A_k \cdot \cos(sx) + B_k \cdot \sin(sx)) e^{-s^2 t}$$

$$u(x, 0) = \sum_{k=0}^{\infty} A_k \cdot \cos(sx) + B_k \cdot \sin(sx) \cdot 1 = f(x)$$

$$F(x, \lambda, u') = u'^2 \cdot f(x)$$

$$L(u) = 0 - \frac{d}{dx} \frac{\delta F}{\delta u'} = 0$$

$$\frac{\delta F}{\delta u'} = c$$

$$\frac{\delta F}{\delta u'} = 2u' \cdot f(x) = c$$

$$u'(x) = \begin{cases} -c \\ c \end{cases}$$

$$0 \leq x < \frac{1}{4}$$

$$\frac{1}{4} \leq x \leq 1$$

$$u(x) = \begin{cases} -cx + d_1 \\ cx + d_2 \end{cases}$$

folytató megoldásokat keressük

$\frac{1}{4}$ -ben összekötni

$\frac{1}{4}$

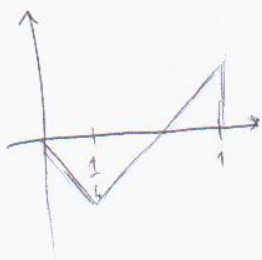
$$d_1 = 0 \leftarrow u(0) = 0$$

$$c + d_2 = 1 \leftarrow u(1) = 1$$

$$-c \frac{1}{4} = c \frac{1}{4} + d_2$$

$$\frac{c}{2} + d_2 = 0$$

$$c + \left(-\frac{c}{2}\right) = 1 \quad c = 2 \quad d_2 = -1 \quad d_1 = 0$$



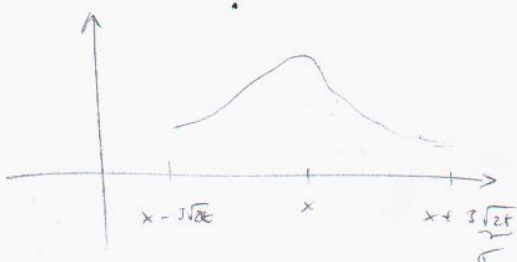
$$u_t = u_{xx} \quad t > 0 \quad x \in \mathbb{R}$$

$$f(x) = u(x, 0) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$u(x, t) = \frac{1}{\sqrt{2\pi} \sqrt{2t}} \int_{-\infty}^{\infty} f(y) \cdot e^{-\frac{(x-y)^2}{4t}} dy$$

$$E(f(S)) \quad S \sim N(x, 2t) \quad \sigma^2$$

$$= \frac{1}{\sqrt{2\pi} \sqrt{2t}} \int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} dy = P(S > 0)$$



$$\int_0^1 x(t) dt \rightarrow \max \quad \int_0^1 \sqrt{1+x'^2(t)} dt = \pi$$

$$I(x, \lambda) = \int_0^1 \left(x(t) - \lambda \sqrt{1+x'^2(t)} \right) dt$$

$$I(u) = \int_{x_0}^{x_1} F(x, u, u') dx$$

$$L[u] = \frac{\delta F}{\delta u} - \frac{d}{dx} \frac{\delta F}{\delta u'} = 0$$

$$F(u, u') = u - \lambda \sqrt{1+u'^2}$$

$$\frac{\delta F}{\delta u} = 1 \quad \frac{\delta F}{\delta u'} = \frac{-u' \lambda}{\sqrt{1+u'^2}} \rightarrow \frac{d}{dx} () \leftarrow \text{Lagrange-Multiplikator}$$

$$\text{Energie: } E(x) = \underbrace{F}_{u - \lambda \sqrt{1+u'^2}} - u' \cdot \frac{\delta F}{\delta u'} = \underbrace{u - \lambda \sqrt{1+u'^2}}_{u - \lambda \sqrt{1+u'^2}} - u' \cdot \frac{-u' \lambda}{\sqrt{1+u'^2}} = C$$

$$u + \frac{\lambda}{\sqrt{1+u'^2}} (-1 + u'^2) = C$$

$$u + \frac{\lambda}{\sqrt{1+u'^2}} = C$$

$$\frac{u-C}{\lambda} = \frac{1}{\sqrt{1+u'^2}}$$

$$\frac{\lambda^2}{(u-C)^2} = 1+u'^2 \quad \frac{du}{dx} = u' = \sqrt{\frac{\lambda^2}{(u-C)^2} - 1}$$

$$\int \frac{1}{\sqrt{\frac{\lambda^2}{(u-C)^2} - 1}} du = x + d$$

8. $u_t = u_{xx} \quad (0,1) \times (0,\infty)$

$u(x,0) = x^2 - x \quad 0 < x < 1$

$u(0,t) = 0 \quad t > 0$

$u(1,t) = 0 \quad t > 0$

$u = T(t) \cdot X(x)$

$u_t = T' \cdot X$

$u_{xx} = T \cdot X''$

$\frac{T'}{T} = \frac{X''}{X} = \lambda_0$

$X(0) \cdot T(t) = 0 \rightarrow X(0) = 0$

$X(1) \cdot T(t) = 0 \rightarrow X(1) = 0$

1) $\lambda_0 = \alpha^2 > 0$

$\frac{T'}{T} = \alpha^2$

$T' - \alpha^2 T = 0$

$T(t) = c_1 \cdot e^{\alpha^2 t}$

$\lambda - \alpha^2 = 0$

$\lambda = \alpha^2$

$\frac{X''}{X} = \alpha^2$

$X'' - \alpha^2 X = 0$

$\lambda^2 - \alpha^2 = 0$

$\lambda^2 = \alpha^2$

$\lambda = \pm \alpha$

$X(x) = c_2 \cdot e^{\alpha x} + c_3 \cdot e^{-\alpha x}$

$X(0) = c_2 + c_3 = 0 \rightarrow c_3 = -c_2$

$X(1) = c_2 e^{\alpha} - c_2 e^{-\alpha} = 0$

$c_2 (e^{\alpha} - e^{-\alpha}) = 0$

$c_2 = 0 \quad ?$

$\alpha = 0 \quad ?$

11) $\lambda_0 = -\alpha^2 < 0$

$\lambda = -\alpha^2 \quad T(t) = c_1 \cdot e^{-\alpha^2 t}$

$T(t) = c_1 \cdot e^{-k^2 \pi^2 t}$

$\lambda = -\alpha^2 \quad X(x) = A \cos(\alpha x) + B \sin(\alpha x)$

$X(0) = A = 0$

$X(1) = B \cdot \sin(\alpha) = 0 \rightarrow \alpha = k\pi$

$B = 0 \quad ?$

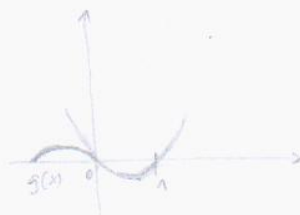
$X_k(x) = B_k \cdot \sin(k\pi x)$

$u(x,t) = \sum_{k=1}^{\infty} B_k \cdot \sin(k\pi x) \cdot c_k \cdot e^{-k^2 \pi^2 t} = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) \cdot e^{-k^2 \pi^2 t}$

$u(x,0) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) = x^2 - x$

$b_k = \int_0^1 g(x) \cdot \sin(k\pi x) dx = 2 \int_0^1 (x^2 - x) \sin(k\pi x) dx$

$b_k = A_k =$ ~~...~~ abzählen



9) ugyanaz, mint a 8. fel

10) $u_t = u_{xx}$ $(0,1) \times (0,\infty)$

$u(x,0) = x^2 - x$ $0 < x < 1$

$u_x(0,t) = 0$ $t > 0$

$u_x(1,t) = 0$ $t > 0$

$u_x(0,t) = X'(0) \cdot T(t) = 0 \rightarrow X'(0) = 0 \rightarrow$

$u_x(1,t) = X'(1) \cdot T(t) = 0 \rightarrow X'(1) = 0$

~~de~~

1) $\lambda_0 = -\alpha^2 < 0$

$\frac{T'}{T} = -\alpha^2$ $T' + \alpha^2 T = 0$

$\lambda + \alpha^2 = 0$

$\lambda = -\alpha^2$

$T(t) = C_1 \cdot e^{-\alpha^2 t}$

$\frac{X''}{X} = -\alpha^2$

$X'' + \alpha^2 X = 0$

$\lambda^2 + \alpha^2 = 0$

$\lambda = \pm i\alpha$

$X(x) = A \cdot \cos(\alpha x) + B \cdot \sin(\alpha x)$

$X'(x) = -A \sin(\alpha x) \cdot \alpha + B \cos(\alpha x) \cdot \alpha$

$X'(0) = B \cdot \alpha = 0$ $\begin{cases} B=0 \\ \alpha=0 \end{cases}$

$X'(1) = -A \sin(\alpha) \cdot \alpha + B \cos(\alpha) \cdot \alpha = 0$

Tff. $B=0$

$-A \sin(\alpha) \alpha = 0$

~~de~~ $A=0$ nem érdekel

$\alpha = k\pi$ $k \in \mathbb{Z}$

Tff. $\alpha=0$

$B \cdot \cos(0) \cdot 0 = 0 \rightarrow$

$0=0$ nem érdekel

$X_k(x) = -Ak\pi \cdot \sin(k\pi x)$

$T_k(t) = C_1 \cdot e^{-k^2\pi^2 t}$

$-Ak\pi \cdot C_1 = B_k$

$u(x,t) = \sum_{k=1}^{\infty} B_k \cdot e^{-k^2\pi^2 t} \cdot \sin(k\pi x)$

$u(x,0) = \sum_{k=1}^{\infty} B_k \cdot \sin(k\pi x) = x^2 - x$

$B_k = B_k = \begin{cases} -\frac{6}{(k\pi)^2} & k \text{ páratlan} \\ 0 & k \text{ páros} \end{cases}$

$u(x,t) = \sum_{k=1}^{\infty} B_k \cdot \sin(k\pi x) \cdot e^{-k^2\pi^2 t}$

ugyanaz a megoldás, csak, az előzőnél ez az drágább?

$$6) \Omega = \{(x, y) \mid x, y > 0\}$$



$$u''_{xx} + u''_{yy} = 0$$

$$u = X(x) \cdot Y(y)$$

$$u(x, 0) = 0 \quad x > 0$$

$$x > 0$$

$$X(x) \cdot Y(0) = 0 \rightarrow Y(0) = 0$$

$$u(0, y) = \sin y \quad y > 0$$

$$\lim_{x \rightarrow \infty} u(x, y) = 0 \quad \forall y > 0$$

$$\lim_{x \rightarrow \infty} X(x) \cdot Y(y) = 0 \rightarrow \lim_{x \rightarrow \infty} X(x) = 0$$

$$\forall y$$

$$X(\infty) = 0$$

$$I) \alpha^2 > 0$$

$$\frac{X''}{X} = \alpha^2$$

$$X'' - \alpha^2 X = 0$$

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$\lambda^2 - \alpha^2 = 0$$

$$\lambda^2 = \alpha^2$$

$$\lambda = \pm \alpha$$

$$X(\infty) = c_1 e^{\alpha \cdot \infty} + c_2 e^{-\alpha \cdot \infty} \rightarrow 0$$

$$\alpha > 0$$

$$c_1 e^{\alpha \cdot \infty} = 0 \Rightarrow c_1 = 0$$

$$X(x) = c_2 e^{-\alpha x}$$

$$II) -\alpha^2 < 0$$

$$\frac{X''}{X} = -\alpha^2$$

$$X'' + \alpha^2 X = 0$$

$$X(x) = A \cos(\alpha x) + B \sin(\alpha x)$$

$$\lambda = \pm \alpha i$$

$$X(\infty) = A \cos(\alpha \cdot \infty) + B \sin(\alpha \cdot \infty) = 0$$

$$Y = A \cdot \cos(\alpha y) + B \cdot \sin(\alpha y)$$

$$Y(0) = A = 0$$

$$Y(y) = B \cdot \sin(\alpha y)$$

$$u(0, y) = X(0) \cdot Y(y) = c_2 e^{-\alpha \cdot 0} \cdot B \cdot \sin(\alpha y) = \sin y$$

$$\rightarrow \alpha = 1$$

$$c_2 \cdot B = 1$$

$$u(x, y) = e^{-x} \cdot \sin y$$

$$\text{ell: } u'_x = -e^{-x} \sin y \quad u''_{xx} = +e^{-x} \sin y$$

$$u'_y = e^{-x} \cdot \cos y \quad u''_{yy} = -e^{-x} \cdot \sin y$$

$$\ominus = 0$$

$$u(x, 0) = e^{-x} \cdot \sin(0) = 0$$

$$\lim_{x \rightarrow \infty} u(x, y) = \lim_{x \rightarrow \infty} e^{-x} \cdot \sin y = 0$$

$$u(0, y) = e^0 \cdot \sin y = \sin y$$

5) $u''_{xx} + u''_{yy} = 0$

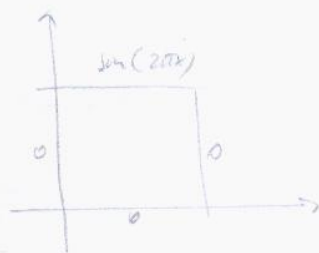
$u(x, 0) = 0 \rightarrow \varphi(0) = 0$

$u(x, 1) = \sin(2\pi x)$

$u(0, y) = 0 \rightarrow \chi(0) = 0$

$u(1, y) = 0 \rightarrow \chi(1) = 0$

$u = \chi(x) \cdot \varphi(y)$



1) $\lambda = -\alpha^2$

$\frac{\chi''}{\chi} = -\alpha^2 \quad \chi'' - \chi\alpha^2 = 0$

$\lambda^2 - \alpha^2 = 0$

$\lambda^2 = \alpha^2$

$\lambda = \pm \alpha$

$\chi(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$\chi(0) = c_1 + c_2 = 0 \rightarrow c_2 = -c_1$

$\chi(1) = 2c_1 \operatorname{sh}(\alpha) = 0$

$\alpha = 0 \rightarrow \gamma$

$c_1 = 0 \rightarrow \gamma$

II) $\lambda = -\alpha^2$

$\frac{\chi''}{\chi} = -\alpha^2 \quad \lambda^2 = -\alpha^2$

$\lambda = \pm \alpha i$

$\chi(x) = A \cos(\alpha x) + B \sin(\alpha x)$

$\chi(0) = A = 0$

$\chi(1) = B \sin(\alpha) = 0 \rightarrow \alpha = k\pi \quad k \in \mathbb{Z}$

$\chi(x) = B \sin(k\pi x)$



$\frac{\varphi''}{\varphi} = \alpha^2 \quad \varphi(y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y}$

$\varphi(0) = c_1 + c_2 = 0 \rightarrow \varphi(y) = 2c_1 \operatorname{sh}(\alpha y) = \underbrace{2c_1}_{=c_2} \operatorname{sh}(k\pi y)$

$u(x, y) = \sum_{k=1}^{\infty} B \cdot c_2 \cdot \sin(k\pi x) \cdot \operatorname{sh}(k\pi y)$

$u(x, 1) = \sum_{k=1}^{\infty} \underbrace{B \cdot c_2}_{A_k} \operatorname{sh}(k\pi) \cdot \sin(k\pi x) = \sin(2\pi x)$

$\left. \begin{array}{l} A_k = 0 \quad \text{ha } k \neq 2 \\ A_k = \frac{1}{\operatorname{sh}(2\pi)} \quad \text{ha } k = 2 \end{array} \right\} \Rightarrow u(x, y) = \frac{1}{\operatorname{sh}(2\pi)} \cdot \sin(2\pi x) \cdot \operatorname{sh}(2\pi y)$

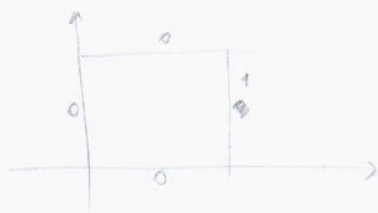
(4) $u_{xx}'' + u_{yy}'' = 0 \quad 0 < x < 1 \quad 0 < y < 1$

$u(x, 0) = 0 \rightarrow Y(0) = 0$

$u(x, 1) = 0 \rightarrow Y(1) = 0$

$u(0, y) = 0 \rightarrow X(0) = 0$

$u(1, y) = 1$



$u(x, y) = X(x) \cdot Y(y)$

$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

1) $\lambda > 0 \quad \lambda = \alpha^2$

$\frac{X''}{X} = \alpha^2 \quad X'' - \alpha^2 X = 0$
 $\lambda^2 - \alpha^2 = 0$
 $\lambda = \pm \alpha$

$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$X(0) = c_1 + c_2 = 0 \rightarrow c_2 = -c_1$

$X(x) = 2c_1 \sinh(\alpha x)$

$\frac{Y''}{Y} = -\alpha^2 \quad Y'' + \alpha^2 Y = 0$
 $\lambda^2 + \alpha^2 = 0$
 $\lambda = \pm \alpha i$

$Y(y) = A \cos(\alpha y) + B \sin(\alpha y)$

$Y(0) = A = 0$

$Y(1) = B \sin(\alpha) = 0 \rightarrow \alpha = k\pi$

$Y(y) = B \cdot \sin(k\pi y)$

$X(x) = 2c_1 \cdot \sinh(k\pi x)$

$u = \sum_{k=1}^{\infty} \overset{A_k}{B \sin(k\pi y)} \cdot \overset{A_k}{2c_1 \sinh(k\pi x)}$

$u(1, y) = \sum_{k=1}^{\infty} \overset{A_k}{2c_1 \sinh(k\pi)} \cdot \overset{A_k}{B \sin(k\pi y)} = 1$

$b_k = 2 \int_0^1 1 \cdot \sin(k\pi x) dx = 2 \left[-\frac{\cos(k\pi x)}{k\pi} \right]_0^1 = \frac{2}{k\pi} (\cos(0) - \cos(k\pi)) = \frac{2}{k\pi} (1 - (-1)^k)$

$A_k = \frac{b_k}{\sinh(k\pi)}$

$u(x, y) = \sum_{k=1}^{\infty} \frac{b_k}{\sinh(k\pi)} \cdot \sin(k\pi y) \cdot \sinh(k\pi x)$

$$u - u' = c \quad u' = u - c$$

$$\frac{du}{dx} = u - c \quad \left| \frac{1}{u-c} du = \int dx = x + d \right.$$

$$\ln(u-c) = x + d$$

$$u - c = e^{x+d}$$

$$u = c_1 \cdot e^x + c_2$$

$$p(0) = 0 = c_1 e^0 + c_2 \rightarrow c_2 = -c_1$$

$$p\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = c_1 e^{\frac{\pi}{2}} - c_1 = c_1 (e^{\frac{\pi}{2}} - 1)$$

$$c_1 = \frac{\frac{\pi}{2}}{e^{\frac{\pi}{2}} - 1}$$

$$p(t) = \frac{\frac{\pi}{2}}{e^{\frac{\pi}{2}} - 1} (e^x - 1) = \frac{\pi}{2} \cdot \frac{e^x - 1}{e^{\frac{\pi}{2}} - 1}$$

$$F = \sqrt{\sin^2 \varphi \cdot \theta'^2 + \rho'^2}$$

$$\frac{\partial F}{\partial \rho} = \frac{\rho'}{\sqrt{\sin^2 \varphi \cdot \theta'^2 + \rho'^2}}$$

$$\sqrt{\sin^2 \varphi \cdot \theta'^2 + \rho'^2} = \frac{\rho'^2}{\sqrt{\sin^2 \varphi \cdot \theta'^2 + \rho'^2}} = c$$

$$\sin^2 \varphi \cdot \theta'^2 + \rho'^2 = \rho'^2 = c \sqrt{\sin^2 \varphi \cdot \theta'^2 + \rho'^2}$$

$$\sin^2 \varphi \cdot \theta'^2 = c \sqrt{\sin^2 \varphi \cdot \theta'^2 + \rho'^2}$$

$$\frac{\sin^2 \varphi \cdot \theta'^4}{c} - \sin^2 \varphi \cdot \theta'^2 = \rho'^2$$

$$\frac{d\varphi}{dt} = \rho' = \sqrt{\frac{\sin^2 \varphi \cdot \theta'^4}{c} - \sin^2 \varphi \cdot \theta'^2}$$

$$\int \frac{1}{\sqrt{\frac{\sin^2 \varphi \cdot \theta'^4}{c} - \sin^2 \varphi \cdot \theta'^2}} d\varphi = \int dt = t + k$$

$$\sqrt{\sin^2 \varphi \cdot \theta'^2 \left(\frac{\sin^2 \varphi \cdot \theta'^2}{c} - 1 \right)}$$

$$\int \frac{1}{\sin \varphi \cdot \theta' \sqrt{\frac{\sin^2 \varphi \cdot \theta'^2}{c} - 1}} d\varphi = \frac{1}{\sin \varphi \cdot \theta' \sqrt{\frac{\sin^2 \varphi \cdot \theta'^2}{c} - 1} \sqrt{\frac{\sin^2 \varphi \cdot \theta'^2}{c} - 1}}$$

① $u_t' + u_x' = 0 \quad t > 0 \quad x \in \mathbb{R}$

$u(x, 0) = e^x = g(x)$

~~$u(x, t) = g(x - bt) = g(x - t) = e^{x-t}$~~

$u_t' = -e^{x-t}$

$u_x' = e^{x-t}$

② $w_t' + w_x' = x - t = f(x, t) \quad | \quad v_t' + v_x' = 0$

$w(x, 0) = e^x = g(x)$

$v(x, 0) = e^x = g(x)$

$v(x, t) = e^{x-t}$

1) $w_t' + w_x' = x - t$

$w(x, 0) = 0$

$w(x, t) = \int_{-t}^0 f(x + bs, t + s) ds = \int_{-t}^0 x + s - t - s ds =$

$= \int_{-t}^0 x - t ds = [xs - ts]_{-t}^0 = -[x(-t) - t(-t)] = +xt - t^2 = w(x, t)$

$u(x, t) = v + w = e^{x-t} + xt - t^2$

$u_t' = -e^{x-t} + x - 2t$

$u_x' = e^{x-t} + t$

$\left. \begin{array}{l} u_t' = -e^{x-t} + x - 2t \\ u_x' = e^{x-t} + t \end{array} \right\} \oplus = x - t \quad \checkmark$

③ $u_t' + 2u_x' = x^2 + 4tx \quad v = 0$

$u(x, 0) = 0$

$w = \int_{-t}^0 (x + 2s)^2 + 4(t + s)(x + 2s) ds = x^2 t = w$

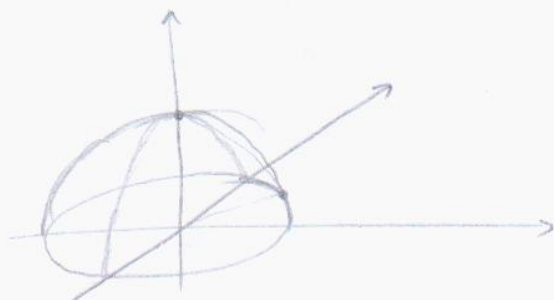
$u_t' = x^2$

$u_x' = 2xt$

$2u_x' = 4xt$

$\left. \begin{array}{l} u_t' = x^2 \\ 2u_x' = 4xt \end{array} \right\} \oplus = x^2 + 4xt$

9) $P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right)$ $P_2 = (0, 0, 1)$
 $\begin{matrix} x & y & z \end{matrix}$



$$x = r \cdot \cos \theta \cdot \sin \varphi = \cos \theta \cdot \sin \varphi$$

$$y = r \cdot \sin \theta \cdot \sin \varphi = \sin \theta \sin \varphi$$

$$z = r \cdot \cos \varphi = \cos \varphi$$

$$r = 1$$

$$(x, y, z) \rightarrow (r, \theta, \varphi) \quad P_1 = \left(1, \frac{\pi}{4}, \frac{\pi}{2} \right)$$

$$P_2 = \left(1, \frac{\pi}{4}, 0 \right)$$



$$\gamma = \left(1, \frac{\pi}{4}, t \right) \quad t \in \left[0, \frac{\pi}{2} \right]$$

nem tudjuk, de látható, hogy az ábrán a
 szögben fog megjelenni.

ez lesz a megoldás polárisban

$$\gamma = \left(\frac{\sqrt{2}}{2} \sin t, \frac{\sqrt{2}}{2} \sin t, \cos t \right) \leftarrow \text{ez meg sima!ban}$$

$$\gamma = (x, y, z) = (\cos \theta(t) \cdot \sin \varphi(t); \sin \theta(t) \cdot \sin \varphi(t); \cos \varphi(t))$$

$$\dot{x}(t) = \sin \varphi \cdot (-\sin \theta) \cdot \theta' + \cos \theta \cdot \cos \varphi \cdot \varphi'$$

$$\dot{y}(t) = \sin \varphi \cdot \cos \theta \cdot \theta' + \sin \theta \cdot \cos \varphi \cdot \varphi'$$

$$\dot{z}(t) = -\sin \varphi \cdot \varphi'$$

$$\dot{x}^2 = \sin^2 \varphi \cdot \sin^2 \theta \cdot \theta'^2 + \cos^2 \theta \cdot \cos^2 \varphi \cdot \varphi'^2 - 2 \cdot \sin \varphi \sin \theta \cdot \cos \theta \cos \varphi \cdot \theta' \cdot \varphi'$$

$$\dot{y}^2 = \sin^2 \varphi \cos^2 \theta \cdot \theta'^2 + \sin^2 \theta \cdot \cos^2 \varphi \cdot \varphi'^2 + 2 \cdot \sin \varphi \sin \theta \cdot \cos \theta \cos \varphi \cdot \theta' \cdot \varphi'$$

$$\dot{z}^2 = \sin^2 \varphi \cdot \varphi'^2$$

$$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \sqrt{\sin^2 \varphi \cdot \theta'^2 + \varphi'^2}$$

$$= \sqrt{\varphi'^2} = \varphi'$$

ha ezt mondjuk, hogy $\theta(t) = \frac{\pi}{4}$

$$\Rightarrow \theta'(t) = 0$$

$$\int_0^{\frac{\pi}{2}} \varphi'(t) dt = I(\varphi)$$

$$\varphi^*(t) \approx u(x)$$

$$F = u(x) \quad \frac{\partial F}{\partial u} = 0 \quad \frac{\partial F}{\partial u'} = 1 \quad \frac{\partial}{\partial x} \frac{\partial F}{\partial u'} = 0$$

$$E(u) = u - u' \cdot 1 = C \quad \rightarrow \quad u' = u + C$$

$$L(u) = 0 - 0 = 0$$

$$1 - u = C \quad u(x) = C_0 \cdot e^x$$

$$k = \frac{\frac{\pi}{2}}{-1 + e^{\frac{\pi}{2}}} = \frac{\frac{\pi}{2}}{1 - e^{\frac{\pi}{2}}}$$

$$k(-1 + e^{\frac{\pi}{2}}) = -\frac{\pi}{2}$$

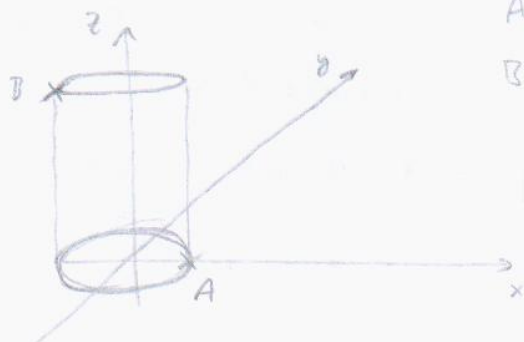
$$\frac{\pi}{2} - k + k e^{\frac{\pi}{2}} = 0$$

$$u(x) = C + k \cdot e^x$$

$$u(0) = \frac{\pi}{2} \rightarrow C + k = \frac{\pi}{2} \quad C = \frac{\pi}{2} - k$$

$$u\left(\frac{\pi}{2}\right) = 0 \rightarrow C + k e^{\frac{\pi}{2}} = 0$$

8.



$A(1, 0, 0) = (1, 0, 0)$

$B(-1, 0, 1) = (1, \pi, 1)$

henger

$x = r \cos \theta$

$r = 1$

$(x, y, z) \rightarrow (r, \theta, z)$

$y = r \sin \theta$

$z = z$

$x(\theta, z) = \cos \theta$

$\theta = \theta(t)$

$y(\theta, z) = \sin \theta$

$z = z(t)$

$z(\theta, z) = z$

általános egyenes

$\gamma(t) = (\cos(\pi t), \sin(\pi t), t)$

$t \in [0, 1]$

$\gamma(t) = (1, \pi t, t)$

henger

Út hossza: $\int_0^1 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$

$z = t$

$x'(t) = (\cos(\theta(t)))' = -\sin(\theta(t)) \cdot \theta'(t)$

$y'(t) = \cos(\theta(t)) \cdot \theta'(t)$

$z'(t) = z'(t)$

$\int_0^1 \sqrt{\sin^2(\theta) \cdot \theta'^2(t) + \cos^2(\theta) \cdot \theta'^2(t) + z'^2(t)} dt$

$\theta'^2(t)$

$z(t) = t$

$z'(t) = 1$

$dt = \int_0^1 \sqrt{1 + \theta'^2(t)} dt$

$\theta(t) \approx u(t)$

$F = \sqrt{1 + u'^2}$

$\frac{\partial F}{\partial u} = 0$

$\frac{\partial F}{\partial u'} = \frac{u'}{\sqrt{1 + u'^2}}$

$E(u) = \sqrt{1 + u'^2} - \frac{u'^2}{\sqrt{1 + u'^2}} = C$

$1 + u'^2 - u'^2 = C \sqrt{1 + u'^2}$

$1 = C \sqrt{1 + u'^2}$

$1 = C(1 + u'^2)$

$\theta(0) = 0 = d$

$\theta(1) = \pi = m$

$\theta(t) = \pi t$

$\frac{1}{C} = k = 1 + u'^2$

$k - 1 = u'^2$

$u' = \pm \sqrt{k - 1}$

$\frac{du}{dx} = \sqrt{k - 1}$

$\int du = \int \sqrt{k - 1} dx$

$u = \sqrt{k - 1} x + d$

$u(x) = mx + d$

$\theta(t) = mt + d$

7. $I(u) = \int_0^1 u^2(x) dx$

a) $u(0) = u(1) = 0$

b) $u(0) = 0, u(1) = 1$

$$F = u^2 \quad \frac{\delta F}{\delta u} = 0 \quad \frac{\delta F}{\delta u'} = 2u \quad \frac{\partial}{\partial x} \frac{\delta F}{\delta u'} = 2u''$$

$$E(u) = u^2 - 2u^2 = c$$

$$L(u) = 0 - 2u'' = 0$$

$$u'' = 0$$

$$u' = c_1$$

$$u = c_1 x + c_2$$

$$-u^2 = c$$

$$u^2 = c$$

$$u' = c$$

$$u = cx + d \quad \text{ugyanaz}$$

a) $u(0) = c_2 = 0 \rightarrow c_2 = 0$

$u(1) = c_1 = 0 \rightarrow c_1 = 0$

$$u(x) \equiv 0$$

b) $u(0) = 0 \rightarrow c_2 = 0$

$u(1) = c_1 \cdot 1 = 1 \rightarrow c_1 = 1$

$$u(x) = x$$

c) $\int_0^1 u(x) dx = 2 \rightarrow \int_0^1 u(x) - 2 dx = 0 \quad G = u(x) - 2$

$$\bar{F} = F - \lambda G = u^2 - \lambda(u-2)$$

$$\frac{\delta \bar{F}}{\delta u} = -\lambda \quad \frac{\delta \bar{F}}{\delta u'} = 2u \quad \frac{\partial}{\partial x} \frac{\delta \bar{F}}{\delta u'} = 2u''$$

$$L(u) = -\lambda - 2u'' = 0$$

$$E(u) = u^2 - \lambda(u-2) - 2u^2 = c$$

$$-\lambda - 2u'' = 0$$

$$2u'' = -\lambda$$

$$u'' = -\frac{\lambda}{2}$$

$$u' = -\frac{\lambda}{2}x + c_1$$

$$u = -\frac{\lambda}{2} \frac{x^2}{2} + c_1 x + c_2$$

$$u(0) = c_2 = 0$$

$$u(1) = -\frac{\lambda}{4} + c_1 = 1$$

$$c_1 = 1 + \frac{\lambda}{4}$$

$$u = -\frac{\lambda}{2} x^2 + \left(1 + \frac{\lambda}{4}\right) x$$

$$\int_0^1 -\frac{\lambda}{2} x^2 + \left(1 + \frac{\lambda}{4}\right) x \, dx = 2$$

$$\left[-\frac{\lambda}{2} \frac{x^3}{3} + \left(1 + \frac{\lambda}{4}\right) \frac{x^2}{2}\right]_0^1 = -\frac{\lambda}{12} + \left(1 + \frac{\lambda}{4}\right) \cdot \frac{1}{2} = -\frac{\lambda}{12} + \frac{1}{2} + \frac{\lambda}{8} = 2$$

$$-\frac{\lambda}{12} + \frac{\lambda}{8} = \frac{3}{2}$$

$$-\frac{2\lambda}{24} + \frac{3\lambda}{24} = \frac{36}{24}$$

$$\lambda = 36$$

$$\underline{u(x) = -9x^2 + 10x}$$

$$\textcircled{5} \quad I(u) = \int_0^1 u'^2 - 4u \, dx$$

$$F = u'^2 - 4u \quad \frac{\partial F}{\partial u} = -4 \quad \frac{\partial F}{\partial u'} = 2u' \quad \frac{d}{dx} \frac{\partial F}{\partial u'} = 2u''$$

$$E(x) = u'^2 - 4u - 2u'^2 = C$$

$$L(u) = -4 - 2u'' = 0$$

$$-2 - u'' = 0$$

$$u'' = -2$$

$$u' = -2x + C_1$$

$$u = -2 \frac{x^2}{2} + C_1 x + C_2$$

$$u = -x^2 + C_1 x + C_2$$

$$\textcircled{6} \quad I(u) = \int_0^\pi u'^2 - 2u^2 \, dx \quad u(0) = 1 \quad u(\pi) = -1$$

$$F = u'^2 - 2u^2 \quad \frac{\partial F}{\partial u} = -4u \quad \frac{\partial F}{\partial u'} = 2u' \quad \frac{d}{dx} \frac{\partial F}{\partial u'} = 2u''$$

$$E(x) = u'^2 - 2u^2 - 2u'^2 = C$$

$$L(u) = -4u - 2u'' = 0$$

$$-2 - \lambda^2 = 0$$

$$\lambda^2 = -2$$

$$\lambda = \pm \sqrt{2} i$$

$$u(x) = A \cos(\sqrt{2} x) + B \sin(\sqrt{2} x)$$

$$u(0) = A = 1$$

$$u(\pi) = \cos(\sqrt{2} \pi) + B \sin(\sqrt{2} \pi) = -1$$

$$B = \frac{-1 - \cos(\sqrt{2} \pi)}{\sin(\sqrt{2} \pi)}$$

$$u(x) = \cos(\sqrt{2} x) + \frac{-1 - \cos(\sqrt{2} \pi)}{\sin(\sqrt{2} \pi)} \sin(\sqrt{2} x)$$

$$\textcircled{3} \quad \int \frac{1}{\sqrt{ku-1}} du = \frac{1}{k} \int \frac{1}{\sqrt{t}} dt = \frac{2}{k} \int \frac{1}{2\sqrt{t}} dt =$$

$$ku-1 = t$$

$$k \frac{du}{dt} = 1 \quad du = \frac{1}{k} dt$$

$$= \frac{2}{k} \sqrt{t} = \frac{2}{k} \sqrt{ku-1}$$

$$\frac{2}{k} \sqrt{ku-1} = x+d$$

$$\frac{4}{k^2} (ku-1) = x^2 + 2xd + d^2 = (x+d)^2$$

$$ku-1 = \frac{k^2}{4} (x+d)^2$$

$$u = \frac{k}{4} (x+d)^2 + \frac{1}{k}$$

$$\textcircled{4} \quad F = u'^2 + 2uu' - 16u^2$$

$$\frac{\partial F}{\partial u} = 2u' - 32u$$

$$\frac{\partial F}{\partial u'} = 2u' + 2u$$

$$\frac{\partial}{\partial x} \frac{\partial F}{\partial u'} = 2u'' + 2u'$$

$$E(x) = u'^2 + 2uu' - 16u^2 - 2u'^2 - 2uu' = c$$

$$-u'^2 - 16u^2 = c$$

$$u'^2 = -16u^2 + c$$

$$u' = \pm \sqrt{-16u^2 + c}$$

$$\frac{du}{dx} = \sqrt{-16u^2 + c}$$

$$\int \frac{1}{\sqrt{-16u^2 + c}} du = \int dx = x+d$$

↖
feilscheit

$$\text{Euler: } 2u' - 32u - 2u'' - 2u' = 0$$

$$-2u'' - 32u = 0$$

$$-u'' - 16u = 0$$

$$- \lambda^2 - 16 = 0$$

$$\lambda^2 = -16$$

$$\lambda = \pm 4i$$

$$u(x) = A \cos(4x) + B \sin(4x)$$

③ $F = \sqrt{u} \sqrt{1+u'^2}$

$$\frac{\partial F}{\partial u} = \sqrt{1+u'^2} \cdot \frac{1}{2\sqrt{u}}$$

$$\frac{\partial F}{\partial u'} = \sqrt{u} \cdot \frac{1}{2\sqrt{1+u'^2}} \cdot 2u' = \sqrt{u} \cdot \frac{u'}{\sqrt{1+u'^2}}$$

$$E(x) = \sqrt{u} \cdot \sqrt{1+u'^2} - \frac{u'^2 \cdot \sqrt{u}}{\sqrt{1+u'^2}} = C \quad / \sqrt{1+u'^2}$$

$$\sqrt{u} (1+u'^2) - u'^2 \sqrt{u} = C \sqrt{1+u'^2}$$

$$\sqrt{u} + \sqrt{u} u'^2 - \sqrt{u} u'^2 = C \sqrt{1+u'^2}$$

$$u = C (1+u'^2) = \text{const}$$

$$\frac{u}{C} = 1+u'^2 \quad u'^2 = \frac{u}{C} - 1 = k \cdot u - 1$$

$$\frac{du}{dx} = u' = \sqrt{k u - 1}$$

$$\int \frac{1}{\sqrt{k u - 1}} du = \int dx = x + d$$

$$\int \frac{1}{\sqrt{k u - 1}} du$$

$$\sqrt{k u} = \text{ch } s$$

$$\text{arch}(\sqrt{k u}) = s$$

$$k u(s) = \text{ch}^2(s)$$

$$u(s) = \frac{\text{ch}^2(s)}{k}$$

$$\frac{du}{ds} = \frac{2 \text{ch}(s) \cdot \text{sh}(s)}{k}$$

$$du = \frac{2}{k} \text{ch}(s) \cdot \text{sh}(s) ds$$

$$\int \frac{1}{\sqrt{k u - 1}} \cdot \frac{2}{k} \text{ch}(s) \cdot \text{sh}(s) ds =$$

$$= \int \frac{1}{\text{sh } s} \cdot \frac{2}{k} \cdot \text{sh}(s) \cdot \text{ch}(s) ds = \frac{2}{k} \int \text{ch}(s) ds = \frac{2}{k} \text{sh}(s) = \frac{2}{k} \cdot \text{sh}(\text{arch}(\sqrt{k u}))$$

$$\frac{2}{k} \text{sh}(\text{arch}(\sqrt{k u})) = x + d$$

$$\text{sh}(\text{arch}(\sqrt{k u})) = \frac{k}{2} (x + d)$$

$$\text{arch}(\sqrt{k u}) = \text{arsh}\left(\frac{k}{2} (x + d)\right)$$

$$\sqrt{k u} = \text{ch}(\text{arsh}(\frac{k}{2} (x + d)))$$

$$k u = \text{ch}^2(\text{arsh}(\frac{k}{2} (x + d)))$$

$$u = \frac{\text{ch}^2(\text{arch}(\frac{k}{2} (x + d)))}{k}$$

2. $F = u^2 + 2u'^2 - 2u \sin x$

$$\frac{\partial F}{\partial u} = 2u - 2 \sin x$$

$$\frac{\partial F}{\partial u'} = 4u'$$

$$\frac{\partial}{\partial x} \frac{\partial F}{\partial u'} = 4u''$$

Euler: $2u - 2 \sin x - 4u'' = 0$

$$2u - 4u'' = 2 \sin x$$

$$u - 2u'' = \sin x$$

1) homogen: $1 - 2\lambda^2 = 0 \quad \lambda = \pm \frac{1}{\sqrt{2}} \rightarrow u(x) = c_1 e^{\frac{1}{\sqrt{2}}x} + c_2 e^{-\frac{1}{\sqrt{2}}x}$
 $\frac{1}{2} = \lambda^2 \rightarrow$

2) inhomogen partikuläre:

$$u = A \cos x + B \sin x$$

$$A \cos x + B \sin x + 2A \cos x + 2B \sin x = \sin x$$

$$u' = -A \sin x + B \cos x$$

$$3A \cos x + 3B \sin x = \sin x$$

$$u'' = -A \cos x - B \sin x$$

$$A = 0 \quad B = \frac{1}{3}$$

$$u_p = \frac{1}{3} \sin x$$

$$u(x) = c_1 e^{\frac{1}{\sqrt{2}}x} + c_2 e^{-\frac{1}{\sqrt{2}}x} + \frac{1}{3} \sin x$$

g) mechanisches System

$$I(u) = \int_0^1 u'^2 \cdot f(x) \, dx \quad f(x) = \begin{cases} -1 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases} \quad \begin{matrix} u(0) = 0 \\ u(1) = 1 \end{matrix}$$

$$F = u'^2 \cdot f \quad \text{Euler: } \frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = 0 \quad \text{Energie: } F - u' \cdot F_{u'} = \text{const}$$

$$\frac{\partial F}{\partial u} = 0 \quad \frac{\partial F}{\partial u'} = 2u' \cdot f \quad \frac{d}{dx} \frac{\partial F}{\partial u'} = 2u'' \cdot f + 2u' \cdot f'$$

$$u'^2 f(x) - u' \cdot 2u' \cdot f = c$$

$$u'^2 f(x) - 2u'^2 f(x) = c$$

$$-u'^2 f(x) = c \quad \text{vergibt}$$

$$+u'^2 = c \quad x \in [0, \frac{1}{2}) \quad -u'^2 = c$$

$$u' = c_2$$

$$u = c_2 \cdot x + d$$

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = 0 \rightarrow 0 - \frac{d}{dx} \frac{\partial F}{\partial u'} = 0 \rightarrow \frac{\partial F}{\partial u'} = c$$

$$2u' f = c_0 \rightarrow u' \cdot f = c$$

$$u' = \begin{cases} -c & x \in [0, \frac{1}{2}) \\ c & x \in [\frac{1}{2}, 1] \end{cases}$$

$$u(x) = \begin{cases} -cx + d_1 \\ cx + d_2 \end{cases}$$

$$u(0) = -c \cdot 0 + d_1 = 0 \rightarrow d_1 = 0$$

$$u(1) = c + d_2 = 1$$

$$-c \cdot \frac{1}{2} = c \cdot \frac{1}{2} + d_2$$

$$0 = c \cdot \frac{1}{2} + d_2$$

$$0 = c \cdot \frac{1}{2} + 1 - c$$

$$+1 = +\frac{1}{2}c$$

$$2 = c$$

$$d_2 = -1$$

① a) $I(u) = \int_0^1 2u(x) + u'(x)^2 dx$ $u(0) = 0$ $u(1) = 1$

$\frac{\delta F}{\delta u} - \frac{d}{dx} \frac{\delta F}{\delta u'} = 0$	$F - u' \cdot F_{u'} = C$
---	---------------------------

← bei nem fests x-toll Variationswert

$$F = 2u + u'^2$$

$$\frac{\delta F}{\delta u'} = 2u'$$

$$E(x) = 2u - u'^2 - u' \cdot 2u' = 2u - 3u'^2 = C$$

Euler: $2 - 2u'' = 0$

$$\frac{d}{dx} \frac{\delta F}{\delta u'} = 2u''$$

$$1 - u'' = 0$$

$$u'' = 1$$

$$\frac{\delta F}{\delta u} = 2$$

$$u' = x + C_1$$

$$u = \frac{x^2}{2} + C_1 x + C_2$$

$$u(0) = \frac{0}{2} + C_1 \cdot 0 + C_2 = 0 \rightarrow C_2 = 0$$

$$u(1) = \frac{1}{2} + C_1 \cdot 1 = 1 \rightarrow C_1 = \frac{1}{2}$$

$$u(x) = \frac{x^2}{2} + \frac{1}{2}$$

b) $F = u'^2 - u^2$ $u(-\frac{\pi}{2}) = u(\frac{\pi}{2}) = 1$

$$\frac{\delta F}{\delta u} = -2u \quad \frac{\delta F}{\delta u'} = 2u' \quad \frac{d}{dx} \frac{\delta F}{\delta u'} = 2u''$$

Euler: $-2u - 2u'' = 0$

$$-1 - \lambda^2 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

$$u(x) = A \cdot \cos(x) + B \cdot \sin(x)$$

$$u(-\frac{\pi}{2}) = B \cdot (-1) = 1 \rightarrow B = -1$$

$$u(\frac{\pi}{2}) = B \cdot 1 = 1 \rightarrow B = 1$$

} ↯

3. (modosult)

$$F = u \cdot \sqrt{1+u'^2}$$

$$\frac{\delta F}{\delta u} = \sqrt{1+u'^2}$$

$$\frac{\delta F}{\delta u'} = u \cdot \frac{u'}{\sqrt{1+u'^2}}$$

$$E(x) = u \cdot \sqrt{1+u'^2} - u \cdot \frac{u'^2}{\sqrt{1+u'^2}} = c \quad / \sqrt{1+u'^2}$$

$$u(1+u'^2) - u \cdot u'^2 = c \sqrt{1+u'^2}$$

$$u + u \cdot u'^2 - u \cdot u'^2 = c \cdot \sqrt{1+u'^2} \quad /^2$$

$$u^2 = c \cdot (1+u'^2)$$

$$\frac{u^2}{c} = 1+u'^2 \quad u'^2 = \frac{u^2}{c} - 1 \quad u' = \sqrt{\frac{u^2}{c} - 1} = \frac{du}{dx}$$

$$\int dx = \int \frac{1}{\sqrt{\frac{u^2}{c} - 1}} du =$$

$$\frac{u^2}{c} - 1 = t(u)$$

$$\frac{2}{c} u$$

$$= \frac{dt}{du}$$

~~desværre~~

$$du = \frac{c}{2u} dt$$

$$= \int \frac{1}{\sqrt{ch^2 t - 1}} \cdot c \cdot sh t dt = \frac{u^2}{c} = \text{~~sh t~~ } ch t \quad t = \operatorname{arsh}\left(\frac{u}{c}\right)$$

$$= \int \frac{1}{sh t} \cdot c \cdot sh t dt = \frac{du}{dt} = \text{~~sh t~~ } c \cdot sh t \quad du = c \cdot sh t dt$$

$$= c \cdot t = c \cdot \operatorname{arsh}\left(\frac{u}{c}\right) = x + d$$

$$\operatorname{arsh}\left(\frac{u}{c}\right) = \frac{x+d}{c}$$

$$\text{~~sh~~ } \frac{u}{c} = ch\left(\frac{x+d}{c}\right)$$

$$u = c \cdot ch\left(\frac{x+d}{c}\right)$$

4) $u_t = u_{xx} \quad x \in \mathbb{R}, t > 0$

Kezdeti körülmények: $u(x, 0) = f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & |x| > 1 \end{cases}$

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_{-\infty}^{\infty} f(y) \cdot e^{-\frac{(y-x)^2}{4t}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_{-1}^1 e^{-\frac{(y-x)^2}{4t}} dy = P(-1 < \xi < 1) = \Phi\left(\frac{1-x}{\sqrt{2t}}\right) - \Phi\left(\frac{-1-x}{\sqrt{2t}}\right)$$

$$\xi \sim N(x, \sqrt{2t}) \quad \frac{\xi-x}{\sqrt{2t}} \sim N(0, 1)$$

Hogyan lehet értelmezni a kapott eredményt?

psz.

11. $u_t = u_{xx}$

$u(x, 0) = \sin(\pi x) \quad 0 < x < 1$; masbhol gaudan 0

$u(0, t) = 0 \quad t > 0$

$u_x(1, t) = 0 \quad t > 0$

nem vigtalen a mid !

$$f(x) = \begin{cases} \sin(\pi x) & x \in \mathbb{R} \cap (0, 1) \\ 0 & \text{masbhol} \end{cases}$$

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_{-\infty}^{\infty} f(y) \cdot e^{-\frac{(y-x)^2}{4t}} dy =$$

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_0^1 \sin(\pi y) \cdot e^{-\frac{(y-x)^2}{4t}} dy$$

$$u(0, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_0^1 \sin(\pi y) \cdot e^{-\frac{y^2}{4t}} dy = 0$$

$$u_x(x, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_0^1 \sin(\pi y) \cdot e^{-\frac{(y-x)^2}{4t}} \cdot [2(y-x) \cdot (-1)] dy \neq 0$$

$$u_x(1, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \int_0^1 \sin(\pi y) \cdot e^{-\frac{(y-1)^2}{4t}} \cdot 2 \cdot (y-1) dy = 0$$

$u(0, t) = u_x(1, t) = 0$

iszonet undariltid parcalis integrallas bilgett biztos van valami eltelmes

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \int_0^1 \sin(\pi y) \cdot e^{-\frac{y^2}{4t}} dy \rightarrow \int \sim N(0, \sqrt{2t})$$

7. dovezetes $u_t = u_{xx}$ Fourier-vál

$$f(x) = u(x, 0) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & |x| > 1 \end{cases}$$

$$\frac{1}{-i} = i$$

$$\frac{1}{i} = -i$$

$$u(x, t) \rightarrow \mathcal{F}(u(x, t), s) = \hat{u}(s, t)$$

$$\mathcal{F}(u_x(x, t), s) = is \hat{u}(s, t)$$

$$\mathcal{F}(u_{xx}(x, t), s) = -s^2 \hat{u}(s, t)$$

$$\frac{d}{dt} \hat{u}(s, t) = -s^2 \hat{u}(s, t)$$

$$\hat{u}(s, t) = e^{-s^2 t} \cdot C(s)$$

$$t=0 \rightarrow \hat{u}(s, 0) = C(s) = \mathcal{F}(u(x, 0), s) = \mathcal{F}(f(x), s)$$

$$\begin{aligned} \mathcal{F}(f(x), s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 f(x) \cdot e^{-isx} dx = \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-isx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-isx}}{-is} \right]_{-1}^1 = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-is} \left[e^{-is} - e^{is} \right] = C(s) \end{aligned}$$

$$\hat{u}(s, t) = e^{-s^2 t} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-is} \left(e^{-is} - e^{is} \right)$$

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}(\hat{u}(s, t), x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(s, t) \cdot e^{isx} ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-is} e^{-s^2 t} \left(e^{-is} - e^{is} \right) e^{isx} ds = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{-is} \right) \cdot e^{-s^2 t - is + isx} - \frac{1}{-is} e^{-s^2 t + is + isx} ds = \\ &= \frac{i}{2\pi} \left[\int_{-\infty}^{\infty} \frac{1}{s} \cdot e^{-s^2 t - is + isx} ds - \int_{-\infty}^{\infty} \frac{1}{s} \cdot e^{-s^2 t + is + isx} ds \right] \end{aligned}$$

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} f(y) \cdot e^{-\frac{(y-x)^2}{4t}} dy = E(f(y)) \quad y \sim N(x, 2t)$$

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \int_{-1}^1 e^{-\frac{(y-x)^2}{4t}} dy = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} \cdot \left(\frac{1}{\sqrt{2t}} \right) \text{ megis}$$

$$(4) \quad u(x,t) = \phi\left(\frac{x}{\sqrt{t}}\right)$$

$$u'_t = \phi'_t\left(\frac{x}{\sqrt{t}}\right) \cdot \left(-\frac{1}{2} \cdot x \cdot t^{-\frac{3}{2}}\right)$$

$$u'_x = \phi'_x\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{1}{\sqrt{t}}$$

$$u''_{xx} = \phi''_{xx}\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{1}{t}$$

$$- \phi'_t\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{x}{2t^{\frac{3}{2}}} = \phi''_{xx}\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{1}{t}$$

$$- \phi'_t\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{x}{\sqrt{t}} = 2 \cdot \phi''_{xx}\left(\frac{x}{\sqrt{t}}\right) \quad u = \frac{x}{\sqrt{t}} \quad \phi\left(\frac{x}{\sqrt{t}}\right) = \phi(u)$$

$$2 \phi''(u) + \phi'(u) u = 0$$

2. 7

$$\frac{1}{\sqrt{2\pi}} \cdot \int_{-1}^1 \frac{1}{\sqrt{2t}} \cdot e^{-\frac{(y-x)^2}{2t}} dy = P(-1 < \tilde{y} < 1) = \Phi\left(\frac{1-x}{\sqrt{2t}}\right) - \Phi\left(\frac{-1-x}{\sqrt{2t}}\right)$$

$$\tilde{y} \sim N(x, \sqrt{2t})$$

↑
notes

$$\frac{\tilde{y} - x}{\sqrt{2t}} \sim N(0, 1)$$

4. $u_t = u''_{xx} \quad x \in \mathbb{R}, t \geq 0$

kezdeti kör: $f(x) = u(x,0) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & |x| > 1 \end{cases}$

$u(t,x) = T(t) \cdot X(x)$ vagy Fourier

$u_t = T' \cdot X \quad u''_{xx} = T \cdot X'' \quad u(x,0) = T(0) \cdot X(x)$

$\frac{T'}{T} = \frac{X''}{X} = \lambda$

1) $\lambda = \alpha^2 > 0$

$T' = \alpha^2 T \quad T' - \alpha^2 T = 0$

$\lambda - \alpha^2 = 0$

$\lambda = \alpha^2$

$T(t) = C_1 e^{\alpha^2 t} = C_1 e^{\lambda t}$

$X'' - \alpha^2 X = 0$

$\lambda^2 - \alpha^2 = 0$

$\lambda^2 = \alpha^2$

$\lambda = \pm \alpha$

$X(x) = C_2 e^{\alpha x} + C_3 e^{-\alpha x}$

$u(x,0) = T(0) \cdot X(x) = C_1 \cdot (C_2 e^{\alpha x} + C_3 e^{-\alpha x}) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & |x| > 1 \end{cases}$

2) $\lambda = -\alpha^2 < 0$

$T' + \alpha^2 T = 0$

$T(t) = C_1 e^{-\alpha^2 t}$

$\lambda = -\alpha^2$

← szumma kéne?

$X'' + \alpha^2 X = 0$

$\lambda^2 = -\alpha^2$

$\lambda = \pm \alpha i$

$X(x) = A \cos(\alpha x) + B \sin(\alpha x)$

$$x = x^2$$

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$$\lambda^2 = \alpha^2$$

$\alpha = 0$

$$e^{ax} + e^{-ax} = 0$$

ii) $d_0 = -x^2 < 0$

$$X(0) = A = 0$$

$$x'(t) = 3 \cdot \alpha \cdot \cos(\alpha) = 0 \rightarrow \alpha = \frac{\pi}{2} + k\pi$$

$$x_1(x) = 8 \sin\left(x\left(\frac{\pi}{2} + 6\pi\right)\right)$$

$b \in \mathbb{H}$

$$T(t) = c_1 \cdot e^{-\left(\frac{\pi}{2} + k\pi\right)^2 t}$$

$$u(x, t) = \sum_{n=0}^{\infty} B_n \sin\left(x\left(\frac{\pi}{2} + n\pi\right)\right) \cdot e^{-\left(\frac{\pi^2}{4} + 4n^2\pi^2\right)t}$$

$$x \frac{\pi}{2} + x k \pi = \pi x (+ k 2\pi)$$

$$x k \pi = \frac{\pi}{2} x$$

$$k\pi = \frac{\pi}{2} \rightarrow k = \frac{1}{2}, \text{ de } k \in \mathbb{Z}$$

11

$$u(x,0) = f(x) = \sin(\pi x) \quad 0 < x < 1$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-x)^2}{4t}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} \cdot \int_0^1 \sin(\pi y) \cdot e^{-\frac{(y-x)^2}{4t}} dy =$$

$$u = e^{-\frac{(y-x)^2}{4t}}$$

$$u' = \sin(\pi y)$$

$$u' = e^{-\frac{(y-x)^2}{4t}} \cdot \left(-\frac{2(y-x)}{4t}\right)$$

$$v = -\frac{\cos(\pi y)}{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} \left[e^{-\frac{(y-x)^2}{4t}} \cdot -\frac{\cos(\pi y)}{\pi} \right]_0^1 - \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} \int_0^1 \frac{+\cos(\pi y)}{\pi} \cdot \frac{+2(y-x)}{2 \cdot 4t} e^{-\frac{(y-x)^2}{4t}} dy =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} \left[\dots \right] - \frac{1}{\pi \cdot 2t \sqrt{2\pi} \sqrt{2t}} \int_0^1$$

$$E(\sin(s)) =$$

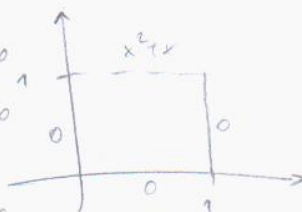
① b) $u''_{xx} + u''_{yy} = 0$

$u(0,y) = 0 \rightarrow X(0) \cdot Y(y) = 0 \rightarrow X(0) = 0$

$u(1,y) = 0 \rightarrow X(1) \cdot Y(y) = 0 \rightarrow X(1) = 0$

$u(x,1) = x^2 + x$

$u(x,0) = 0 \rightarrow X(x) \cdot Y(0) = 0 \rightarrow Y(0) = 0$



$u(x,y) = X(x) \cdot Y(y)$

$$\left. \begin{aligned} u''_{xx} &= X''(x) \cdot Y(y) \\ u''_{yy} &= Y''(y) \cdot X(x) \end{aligned} \right\} X'' \cdot Y + Y'' \cdot X = 0 \quad \frac{X''}{X} = - \frac{Y''}{Y} = \lambda$$

I) $\lambda > 0 \quad \lambda = \alpha^2$

$\frac{X''}{X} = \alpha^2 \quad X'' - \alpha^2 X = 0$

$\lambda^2 - \alpha^2 = 0$

$\lambda^2 = \alpha^2$

$\lambda = \pm \alpha$

$X(x) = c_1 e^{+\alpha x} + c_2 e^{-\alpha x}$

$X(0) = c_1 + c_2 \rightarrow c_2 = -c_1$

$X(1) = c_1 e^{\alpha} - c_1 e^{-\alpha} = 2c_1 \operatorname{sh}(\alpha) = 0$

$\operatorname{sh}(\alpha) = 0 \rightarrow \alpha = 0$

II) $\lambda < 0 \quad \lambda = -\alpha^2$

$\frac{X''}{X} = -\alpha^2 \quad X'' + \alpha^2 X = 0$

$\lambda^2 + \alpha^2 = 0$

$\lambda = \pm \alpha i$

$X(x) = A \cos(\alpha x) + B \sin(\alpha x)$

$X(0) = A = 0$

$X(1) = B \cdot \sin(\alpha) = 0 \quad \alpha = k\pi$

$\frac{Y''}{Y} = -\alpha^2 \quad Y(y) = c_1 e^{+\alpha y} + c_2 e^{-\alpha y}$

$Y(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$Y(y) = 2c_1 \operatorname{sh}(\alpha y) = 2c_1 \operatorname{sh}(k\pi y)$

$Y(1) \cdot X(x) = x^2 + x \rightarrow 2c_1 \operatorname{sh}(k\pi) \cdot B \sin(k\pi x) = x^2 + x$

$u(x,y) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) \cdot \operatorname{sh}(k\pi y)$

$u(x,1) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) \cdot \operatorname{sh}(k\pi) = x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$



$$b_k = \frac{2}{T_0} \int_0^1 (x^2+x) \sin\left(\frac{k\pi x}{T_0}\right) dx = 2 \int_0^1 x^2 \cdot \sin(k\pi x) dx + 2 \int_0^1 x \cdot \sin(k\pi x) dx =$$

$$u = x^2+x \quad v' = \sin(k\pi x)$$

$$u' = 2x+1 \quad v = \frac{-\cos(k\pi x)}{k\pi}$$

$$= 2 \left[\left[(x^2+x) \left(-\frac{\cos(k\pi x)}{k\pi} \right) \right]_0^1 + \int_0^1 (2x+1) \left(\frac{\cos(k\pi x)}{k\pi} \right) dx \right] =$$

$$= 2 \left[\frac{2}{k\pi} (-\cos(k\pi)) + \left[(2x+1) \frac{\sin(k\pi x)}{(k\pi)^2} \right]_0^1 - \int_0^1 2 \frac{\sin(k\pi x)}{(k\pi)^2} dx \right] =$$

$$= \frac{-4}{k\pi} \cos(k\pi) + 4 \left[\frac{\cos(k\pi)}{(k\pi)^2} \right]_0^1 = \frac{-4}{k\pi} \cos(k\pi) + 4 \left[\frac{\cos(k\pi)}{(k\pi)^2} - \frac{1}{(k\pi)^2} \right] =$$

$$= (-1) \frac{4}{k\pi} (-1)^k + 4 \frac{(-1)^k}{(k\pi)^2} - \frac{4}{(k\pi)^2} = b_k$$

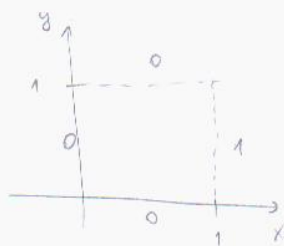
$$x^2+x = \sum_{k=1}^{\infty} b_k \cdot \sin(k\pi x) \rightarrow A_k \cdot \sin(k\pi) = b_k \rightarrow A_k = \frac{b_k}{\sin(k\pi)}$$

$$u(x,y) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) \cdot \sin(k\pi y)$$

②

① $u''_{xx} + u''_{yy} = 0$ $0 < x < 1$, $0 < y < 1$

a)



$$\begin{aligned} u(0,y) &= 0 & x(0) \cdot Y(y) &= 0 \Rightarrow x(0) = 0 \\ u(1,y) &= 1 & x(1) \cdot Y(y) &= 1 \\ u(x,1) &= 0 & x(x) \cdot Y(1) &= 0 \Rightarrow Y(1) = 0 \\ u(x,0) &= 0 & x(x) \cdot Y(0) &= 0 \Rightarrow Y(0) = 0 \end{aligned}$$

Tf. $u(x,y) = X(x) \cdot Y(y)$

$$\left. \begin{aligned} u''_{xx} &= Y(y) \cdot X''(x) \\ u''_{yy} &= X(x) \cdot Y''(y) \end{aligned} \right\} \rightarrow \Delta u = X''(x) \cdot Y(y) + X(x) \cdot Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = - \frac{Y''(y)}{Y(y)} = \lambda$$

b) Tf. $\lambda = \alpha^2$ then $\lambda > 0$

$$\frac{X''(x)}{X(x)} = \alpha^2$$

$$X''(x) - \alpha^2 X(x) = 0$$

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$\lambda^2 - \alpha^2 = 0$$

$$\lambda = \pm \alpha$$

$$X(0) = c_1 + c_2 = 0 \rightarrow c_2 = -c_1$$

$$X(x) = c_1 e^{\alpha x} - c_1 e^{-\alpha x} = 2c_1 \cdot \sinh(\alpha x)$$

$$\frac{Y''(y)}{Y(y)} = -\alpha^2$$

$$Y'' + \alpha^2 Y = 0$$

$$\lambda^2 + \alpha^2 = 0$$

$$\lambda = \pm i\alpha$$

$$Y(y) = A \cos(\alpha y) + B \sin(\alpha y)$$

$$Y(0) = A \cdot 1 + B \cdot 0 \rightarrow A = 0$$

$$Y(1) = B \cdot \sin(\alpha) = 0$$

$$B \cdot \sin(\alpha) = 0$$

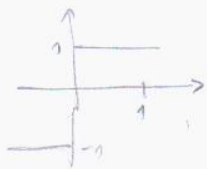
$$\alpha = k \cdot \pi \quad k \in \mathbb{Z}$$

$$X(x) = 2c_1 \sinh(k\pi \cdot x)$$

$$u(x,y) = 2c_1 \sinh(k\pi x) \cdot B \sin(k\pi y) = A_k \sinh(k\pi x) \sin(k\pi y)$$

$$u(1,y) = 2c_1 \sinh(k\pi) \cdot B \sin(k\pi y) = 1$$

$$A_k \sinh(k\pi) \sin(k\pi y) = 1$$



$$b_k = 2 \int_0^1 1 \cdot \sin(k\pi y) dy = 2 \left[\frac{-\cos(k\pi y)}{k\pi} \right]_0^1 =$$

$$= \frac{-2}{k\pi} [\cos(k\pi) - 1] = -\frac{2}{k\pi} \cdot ((-1)^k - 1)$$

cos k ist gerade
ungerade

$$\rightarrow a_k = 2 \int_0^1 \cos(k\pi y) dy = 2 \left[\frac{\sin(k\pi y)}{k\pi} \right]_0^1 = 0 \checkmark$$

$$A_k \cdot \sin(k\pi) = b_k$$

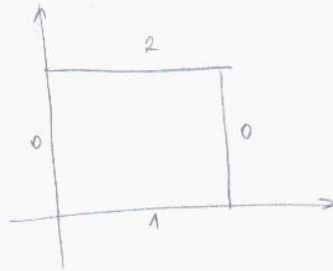
$$A_k = \frac{b_k}{\sin(k\pi)}$$

$$u(x, y) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi y) \cdot \sin(k\pi x)$$



2) HF

$$u''_{xx} + u''_{yy} = 0$$



$$\begin{cases} X(x) \cdot Y(0) = 1 \\ X(x) \cdot Y(2) = 2 \end{cases} \Rightarrow Y(1) = 2 \cdot Y(0)$$

$$X(0) \cdot Y(y) = 0 \Rightarrow X(0) = 0$$

$$X(1) \cdot Y(y) = 0 \Rightarrow X(1) = 0$$

1) Tfl $\lambda > 0$ $\lambda = \alpha^2$

$$\frac{\lambda''}{\lambda} = \lambda = \alpha^2 \quad \begin{aligned} X'' - \alpha^2 X &= 0 \\ \lambda^2 - \alpha^2 &= 0 \\ \lambda &= \pm \alpha \end{aligned}$$

$$X(x) = c_1 \cdot e^{\alpha x} + c_2 \cdot e^{-\alpha x}$$

$$X(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$X(1) = 2c_1 \sinh(\alpha) = 0$$

$\alpha = 0 \Rightarrow$ nem jó választás

2) Tfl $\lambda < 0$ $\lambda = -\alpha^2$

$$X'' + \alpha^2 X = 0$$

$$X(x) = A \cos(\alpha x) + B \sin(\alpha x)$$

$$\lambda^2 + \alpha^2 = 0$$

$$X(0) = A = 0$$

$$\lambda = \pm \alpha$$

$$X(1) = B \sin(\alpha) = 0 \Rightarrow \alpha = k\pi \quad k \in \mathbb{Z}$$

$$\frac{Y''}{Y} = \alpha^2 \Rightarrow Y(y) = 2c_1 \sinh(\alpha y) = 2c_1 \sinh(k\pi y)$$

$$Y(y) = c_1 e^{k\pi y} + c_2 e^{-k\pi y}$$

$$c_1 e^{k\pi} + c_2 e^{-k\pi} = 2(c_1 + c_2)$$

$$c_1 e^{k\pi} + c_2 e^{-k\pi} = 2c_1 + 2c_2$$

szuperpozíció: 1) $V(x, 1) = 2$
 $V(x, 0) = 0$

↑
nem

$$c_1 (e^{k\pi} - 2) = c_2 (2 - e^{-k\pi})$$

művelet

$$\frac{c_1}{c_2} = \frac{2 - e^{-k\pi}}{e^{k\pi} - 2}$$

de itt nem jó választás

$$V(x, y) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) \sinh(k\pi y)$$

$$V(x, 1) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) \sinh(k\pi) = 2 \Rightarrow A_k \sinh(k\pi) = b_k$$

$$b_k = 2 \int_0^1 2 \sin(k\pi x) dx = -4 \left[\frac{\cos(k\pi x)}{k\pi} \right]_0^1 = \frac{-4}{k\pi} (\cos(k\pi) - 1) = \frac{-4}{k\pi} ((-1)^k - 1)$$

$$A_k = \frac{b_k}{\sinh(k\pi)}$$

2) $W(x, 1) = 0$
 $W(x, 0) = 1$

$$W(x, y) = \sum_{k=1}^{\infty} B_k \sin(k\pi x) \sinh(k\pi y)$$

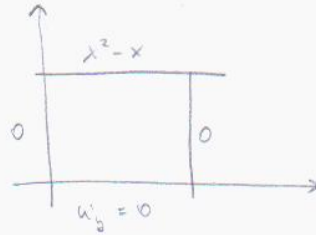
$$W(x, 0) = \sum_{k=1}^{\infty} B_k \sin(k\pi x) \underbrace{\sinh(0)}_0 = 1 \Rightarrow \text{nem lehet, mert az } \sinh(0) \text{ nulla'ra az egész}$$

3. $u_y(x, 0) = 0$

$u(x, 1) = x^2 - x$

$u(0, y) = 0$

$u(1, y) = 0$



TfL: $u(x, y) = X(x) \cdot Y(y)$

$u_y(x, 0) = X(x) \cdot Y'(0) \rightarrow Y'(0) = 0$

$X(x) \cdot Y(1) = x^2 - x$

$X(0) \cdot Y(y) = 0 \rightarrow X(0) = 0$

$X(1) \cdot Y(y) = 0 \rightarrow X(1) = 0$

1.) TfL: $\lambda = \alpha^2 > 0$

$\frac{X''}{X} = \alpha^2 \quad X'' = \alpha^2 X$
 $X'' - \alpha^2 X = 0$

$\lambda^2 - \alpha^2 = 0$

$\lambda^2 = \alpha^2$

$\lambda = \pm \alpha$

$X(x) = c_1 \cdot e^{\alpha x} + c_2 \cdot e^{-\alpha x}$

$X(0) = c_1 + c_2 = 0 \rightarrow c_2 = -c_1$

$X(1) = 2c_1 \sinh(\alpha) = 0 \rightarrow c_1 = 0$

$\rightarrow \alpha = 0$

2.) TfL: $\lambda = -\alpha^2 > 0$

$\frac{X''}{X} = -\alpha^2 \quad \lambda = \pm \alpha i$

$X(x) = A \cdot \cos(\alpha x) + B \cdot \sin(\alpha x)$

$X(0) = A = 0$

$X(1) = B \cdot \sin(\alpha) = 0 \rightarrow \alpha = k\pi \quad k \in \mathbb{Z}$

$X(x) = B \cdot \sin(k\pi x)$

$\frac{Y''}{Y} = \alpha^2 \quad Y(y) = c_1 \cdot e^{\alpha y} + c_2 \cdot e^{-\alpha y}$

$Y(y) = c_1 \cdot e^{k\pi y} + c_2 \cdot e^{-k\pi y}$

$Y'(y) = k\pi c_1 e^{k\pi y} - k\pi c_2 e^{-k\pi y}$

$Y'(0) = k\pi c_1 - k\pi c_2 = 0 \rightarrow c_1 = c_2$

$Y(y) = c_1 e^{k\pi y} + c_1 e^{-k\pi y} = 2c_1 \cdot \cosh(k\pi y) = A \cdot \cosh(k\pi y)$

$u(x, y) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) \cdot \cosh(k\pi y)$

$u(x, 1) = \sum_{k=1}^{\infty} A_k \cdot \cosh(k\pi) \cdot \sin(k\pi x) \rightarrow A_k \cdot \cosh(k\pi) = b_k$

$$b_k = \textcircled{2} \int_0^1 (x^2 - x) \sin(k\pi x) dx = \frac{4}{(k\pi)^3} [(-1)^k - 1]$$

folgen-
 oder nicht!

$$u = x^2 - x \quad v' = \sin(k\pi x)$$

$$u' = 2x - 1 \quad v = \frac{-\cos(k\pi x)}{k\pi}$$

$$\left[(x^2 - x) \frac{-\cos(k\pi x)}{k\pi} \right]_0^1 + \int_0^1 (2x - 1) \frac{\cos(k\pi x)}{k\pi} dx = \underbrace{\left[(2x - 1) \frac{\sin(k\pi x)}{(k\pi)^2} \right]_0^1}_{=0} - \int_0^1 2 \frac{\sin(k\pi x)}{(k\pi)^2} dx$$

①
 0

$$u = 2x - 1 \quad v' = \frac{\cos(k\pi x)}{k\pi}$$

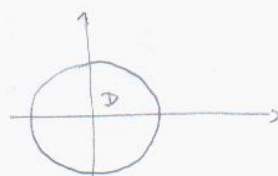
$$u' = 2 \quad v = \frac{\sin(k\pi x)}{(k\pi)^2}$$

$$= +2 \left[\frac{\sin(k\pi x)}{(k\pi)^3} \right]_0^1 = \frac{2}{(k\pi)^3} [(-1)^k - 1]$$

$$A_k = \frac{b_k}{\cosh(k\pi)}$$

$$u(x, y) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) \cdot \cosh(k\pi y)$$

4. $D = \{(x, y) : x^2 + y^2 < 1\}$



$$u''_{xx} + u''_{yy} = 0$$

$$x = r \cdot \cos \theta$$

$$u(r, \theta) = u(r \cos \theta, r \sin \theta)$$

$$r \in [0, 1)$$

$$y = r \cdot \sin \theta$$

$$\theta \in [0, 2\pi)$$

$$\frac{\partial x}{\partial r} = \cos \theta$$

$$\frac{\partial x}{\partial \theta} = r \cdot (-\sin \theta)$$

$$\frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial y}{\partial \theta} = r \cdot \cos \theta$$

→

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r} \frac{\partial u}{\partial r} = \cos \theta \cdot \frac{\partial}{\partial r} \frac{\partial u}{\partial r} + \sin \theta \cdot \frac{\partial}{\partial r} \frac{\partial u}{\partial y}$$

$$= \cos \theta \left(\frac{\partial}{\partial x} \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \frac{\partial u}{\partial x} \frac{\partial y}{\partial r} \right) + \sin \theta \left(\frac{\partial}{\partial x} \frac{\partial u}{\partial y} \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \right) =$$

$$= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2} =$$

$$= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta)$$

$$\frac{\partial}{\partial \theta} \frac{\partial u}{\partial r} = -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial}{\partial \theta} \frac{\partial u}{\partial x} + -r \sin \theta \frac{\partial u}{\partial y} + r \cos \theta \frac{\partial}{\partial \theta} \frac{\partial u}{\partial y} =$$

$$= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \left(\frac{\partial}{\partial x} \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \frac{\partial u}{\partial x} \frac{\partial y}{\partial \theta} \right) + \left(\frac{\partial}{\partial x} \frac{\partial u}{\partial y} \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} \right) + r \cos \theta \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \right) + \frac{\partial^2 u}{\partial y^2} r \cos \theta$$

$$= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \left(\frac{\partial^2 u}{\partial x^2} (-r \sin \theta) + \left(\frac{\partial^2 u}{\partial x \partial y} \right) (r \cos \theta) \right) + \left(\frac{\partial^2 u}{\partial x \partial y} (-r \sin \theta) + \frac{\partial^2 u}{\partial y^2} (r \cos \theta) \right) + \frac{\partial^2 u}{\partial x^2} r \cos \theta + \frac{\partial^2 u}{\partial x \partial y} 2 \sin \theta \cos \theta + \cos^2 \theta \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2} r \cos \theta + \frac{\partial^2 u}{\partial x \partial y} 2 \sin \theta \cos \theta + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \quad (2)$$

$$\frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r} \frac{\partial u}{\partial x} + \sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \sin \theta \cos \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \quad (2)$$

① + ②

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

↓

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$u''_{xx} + u''_{yy} = u''_{rr} + \frac{1}{r} u'_r + \frac{1}{r^2} u''_{\theta\theta} = 0$$

$$u_t'(t,x) + 2 \cdot u_x'(t,x) = x^2 + 4tx = f(t,x)$$

$$u(0,x) = 0 = g(x)$$

$$u(t,x) = u_1(t,x) + u_2(t,x)$$

1) homogen: $u_1_t' + 2 \cdot u_1_x' = 0$ $b=2$

$$u_1(0,x) = 0 = g(x)$$

$$u_1(t,x) = g(x - bt) = 0$$

II) inhomogen: $u_2_t' + 2 u_2_x' = x^2 + 4tx = f$

$$u_2(0,x) = 0$$

$$u_2(t,x) = \int_{-t}^0 f(x+bs, t+s) ds = \int_{-t}^0 (x+bs)^2 + 4(t+s)(x+bs) ds =$$

$$u_t' + b \cdot u_x' = f(x,t)$$

$$u(t,x) = v(t,x) + w(t,x)$$

$$u(0,x) = g(x)$$

$$v(t,x) = g(x - bt)$$

$$w(t,x) = \int_{-t}^0 f(x+bs, t+s) ds$$

⑥ $u_t'(t,x) + u_x'(t,x) = x - t = f(t,x)$

$$u(0,x) = e^x = g(x)$$

1) $v_t' + v_x' = 0$
 $v(0,x) = e^x$

$$v = g(x - bt) = e^{x-t}$$

II) $w_t' + w_x' = x - t$
 $w(0,x) = 0$

$$w = \int_{-t}^0 f(x+bs, t+s) ds = \int_{-t}^0 (x+s) - (t+s) ds = \int_{-t}^0 x - t ds =$$

$$= (x-t)s \Big|_{-t}^0 = +t(x-t)$$

$$w = e^{x-t} + t(x-t) = e^{x-t} + tx - t^2$$

$$\left. \begin{aligned} u_t' &= -e^{x-t} + x - 2t \\ u_x' &= e^{x-t} + t \end{aligned} \right\} \oplus \rightarrow x - t \checkmark$$

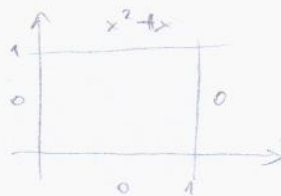
$$\Delta u = u''_{xx} + u''_{yy} = 0$$

$$\text{Ist } u = X(x) + Y(y)$$

$$u''_{xx} = Y(y) \cdot X''(x)$$

$$u''_{yy} = Y''(y) \cdot X$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$



$$u(x, 0) = 0 = \cancel{x^2 + 1} X(x) \cdot Y(0) \rightarrow Y(0) = 0$$

$$u(x, 1) = x^2 + 1$$

$$u(0, y) = 0 = X(0) \cdot Y(y) \rightarrow X(0) = 0$$

$$u(1, y) = 0 = X(1) \cdot Y(y) \rightarrow X(1) = 0$$

$$(5) \quad u'_t(t, x) + u'_x(t, x) = 0 \quad t > 0 \quad x \in \mathbb{R}$$

$$u(0, x) = e^x = g(x)$$

$$b = 1 \quad u(x, t) = g(x - bt) = g(x - t) = e^{x-t}$$

$$u'_t = -e^{x-t} \quad u'_x = e^{x-t}$$

$$(6) \quad u'_t(t, x) + u'_x(t, x) = x - t$$

$$u(0, x) = e^x = g(x)$$

$$I) \text{ homogena: } v'_t + b \cdot v'_x = 0$$

$$\rightarrow v(x, t) = e^{x-t}$$

$$v(x, 0) = g(x) = e^x$$

$$II) \text{ inhomogena: } w'_t + b \cdot w'_x = f(x, t) = x - t$$

$$w(x, 0) = 0$$

↓

$$w(x, t) = \int_{-t}^0 f(x + bs, t-s) ds = \int_{-t}^0 (x - t) ds = \left[xs - ts \right]_{-t}^0 = 0 - (-t)(x - t) = t(x - t)$$

$$u(x, t) = e^{x-t} + t(x-t)$$

$$u'_t = -e^{x-t} + x - 2t$$

$$u'_x = e^{x-t} + t$$

$$\left. \begin{array}{l} u'_t = -e^{x-t} + x - 2t \\ u'_x = e^{x-t} + t \end{array} \right\} \oplus \quad x - t \quad \checkmark$$

$$4) \quad u_t'(t, x) + 2 u_x'(t, x) = x^2 + 4tx$$

$$u(0, x) = 0 = g(x)$$

1) homogen: $v_t'(t, x) + 2 v_x'(t, x) = 0$

$$b=2$$

$$v(0, x) = 0$$

$$v(x, t) = g(x - 2t) = 0$$

11) $w_t'(t, x) + 2 w_x'(t, x) = x^2 + 4tx$

$$w(x, 0) = 0$$

$$w(x, t) = \int_{-t}^0 f(x+bs, t+s) ds = \int_{-t}^0 (x+2s)^2 + 4(t+s)(x+2s) ds =$$

$$= \int_{-t}^0 x^2 + 4xs + 4s^2 + 4tx + 4xs + 8st + 8s^2 ds =$$

$$= \left[x^2 s + 4x \frac{s^2}{2} + 4 \frac{s^3}{3} + 4txs + 4x \frac{s^2}{2} + 8t \frac{s^2}{2} + 8 \frac{s^3}{3} \right]_{-t}^0 =$$

$$= - \left[x^2(-t) + 2xt^2 - \frac{4}{3}t^3 - 4xt^2 + 2xt^2 + 4t^3 - \frac{8}{3}t^3 \right] =$$

$$= x^2 t - 2xt^2 + \frac{4}{3}t^3 + 4xt^2 - 2xt^2 - 4t^3 + \frac{8}{3}t^3 = x^2 t$$

$$u_t' = x^2 - \cancel{4xt} + \cancel{4t^2} + \cancel{8xt} - \cancel{4xt} - \cancel{12t^2} + \cancel{8t^2} = x^2$$

$$u_x' = 2xt - 2t^2 + 4t^2 - 2t^2 = 2xt$$

$$u_t' + 2 u_x' = x^2 + 4xt \quad \checkmark$$