Komplex fv tan (kugró)

(Kozzi Vârdai)

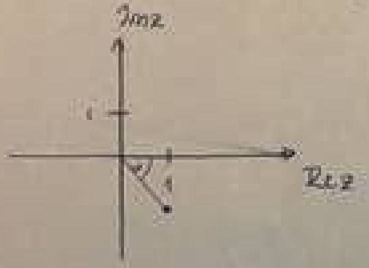
$f(z) = u(x,y) + i v(x,y)$ → kanonikus alak

↓

nem hagyunk a nevezőben i-t

$f(z) = \dfrac{1+i}{1-2i} \rightarrow \dfrac{1+2i}{1+2i} = \dfrac{1-2}{1+4} + \dfrac{i+2i}{5} = -\dfrac{1}{5} + i\dfrac{3}{5}$ → $f(z)$ kanonikus alak

$z\,\bar{z} = a^2 + b^2$



$a+ib = r(\cos\alpha + i\sin\alpha)$

$r = \sqrt{a^2+b^2}$

$\underbrace{\operatorname{tg}\alpha}_{\frac{b}{a}} = -1 \qquad \alpha = \dfrac{7\pi}{4}$

$1-i = \sqrt{2}\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$

$= e^{i\,7\pi/4}\cdot\sqrt{2}$

$f(z) = \bar{z} = x - iy$ → diffható-e? (Biz.)

Bizonyítás: $\left.\begin{array}{l} u'_x = v'_y \\ u'_y = -v'_x \end{array}\right\}$ nem teljesül → nem diffható

$f(z) = \dfrac{1+z}{1-z}$ diffható-e?

$\dfrac{1+x+iy}{1-x-iy}\cdot\dfrac{1-x+iy}{1-x+iy} = \dfrac{(1+iy)^2 - x^2}{(1-x)^2+y^2} =$

$= \dfrac{1+2iy-y^2-x^2}{(1-x)^2+y^2} = \boxed{\underbrace{\dfrac{1-y^2-x^2}{(1-x)^2+y^2}}_{u} + i\underbrace{\dfrac{2y}{(1-x)^2+y^2}}_{v}}$

$u'_x = \dfrac{-2x\left((1-x)^2+y^2\right) - (1-y^2-x^2)\,2(1-x)}{\left((1-x)^2+y^2\right)^2}$

$= \dfrac{-2x(1-x)^2 - 2xy^2 + 2 - 2y^2 - 2x^2 - 2x + 2xy^2 + 2x^3}{}$

$1-2x+x^2$

$-2x+4x^2-2x^3 -$

$-4x + 2x^2 + 2 - 2y^2 = 2(1-x^2) - 2y^2$

$v'_y = \dfrac{2\left((1-x)^2+y^2\right) - 2y\cdot 2y}{\left((1-x)^2+y^2\right)^2}$

$= \dfrac{2(1-x)^2 - 2y^2}{\left((1-x)^2+y^2\right)^2}$

⇓

Tehát $u'_x = v'_y$ ✓ → Ellenőrizzük a 2. egyenletet is

$u'_y = -v'_x$

$u'_y = \dfrac{-2y(1-x)^2 - 2y(1-y^2-x^2)\,2y}{\left((1-x)^2+y^2\right)^2} = \dfrac{-2y+4xy-2yx^2-2y^3-2y+2y^3+2x^2y}{\left((1-x)^2+y^2\right)^2} = \dfrac{-4y+4xy}{\left((1-x)^2+y^2\right)^2} =$

$\dfrac{-4y(1-x)}{\left((1-x)^2+y^2\right)^2}$

$v'_x = \dfrac{0 - 2y\cdot 2(1-x)}{\left((1-x)^2+y^2\right)^2} = \dfrac{4y(1-x)}{\left((1-x)^2+y^2\right)^2}$

⇓

$-v'_x = u'_y$

$$f'(z) = u'_x + i\,v'_x = \frac{2(1-x)^2 - 2y^2}{((1-x)^2+y^2)^2} + i\,\frac{-4y(1-x)}{((1-x)^2+y^2)^2}$$

$$= v'_y - i\,u'_y$$

Differenciálható-e?

$f(z) = \overline{z}^3 = (\overline{z})^3 \quad \overline{z} = \overline{x+iy} = x - iy \rightarrow (x-iy)(x-iy)(x-iy)$

$(x^2 - 2ixy - y^2)(x-iy) = x^3 - 3ix^2y - 3xy^2 + iy^3 =$

$\underbrace{x^3 - 3xy^2}_{u} + i\underbrace{\left[-3x^2y + y^3\right]}_{v}$

$u'_x = 3x^2 - 3y^2$

$u'_y = -6xy$

$v'_y = -3x^2 + 3y^2$

Harmonikus társ keresés

$u(x,y) = x^2 - y^2 + xy$

u harmonikus, ha $u''_{xx} + u''_{yy} = 0$

$u'_x = 2x + y \qquad u'_y = -2y + x$

$u''_{xx} = 2 \qquad u''_{yy} = -2$

$u'_x = v'_y \qquad u'_y = -v'_x$

$v = \int u'_x \, dy = \int 2x + y \, dy = 2xy + \frac{y^2}{2} + g(x)$

$v'_x = 2y - x \rightarrow v = \int -u'_y \, dx = \int 2y - x \, dy = 2xy - \frac{x^2}{2} + h(y)$

$\Bigg\} \; v(x,y) = 2xy + \frac{y^2}{2} - \frac{x^2}{2}$

$f(z) = x^2 - y^2 + xy + i\left(2xy + \frac{y^2}{2} - \frac{x^2}{2}\right) \Rightarrow$ harmonikus fv-ből állítottuk össze

↳ tehát diffható $f(z)$

Komplex fv-ek vonalintegrálja

$f(z) = z^2 \qquad a = i \qquad b = -i$

$\int_\Gamma f(\gamma(t)) \cdot \gamma'(t) \, dt$

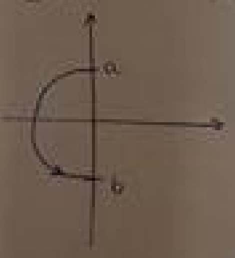
$\gamma(t) = i(1-t) \qquad t \in [0,2] \qquad t = 0 \rightarrow a, \; t = 2 \rightarrow b$

$\gamma'(t) = -i$

$$\int_\Gamma f(\gamma(t)) \cdot \gamma'(t) \, dt = \int_0^2 [i(t-1)]^2 \cdot (-i) \, dt = \int_0^2 i(1-t)^2 \, dt = i\,\frac{(1-t)^3}{3(-1)}\Bigg|_0^2 = i\left[\frac{1}{3} + \frac{1}{3}\right] = \frac{2}{3}i$$

$f_2(z) = z^2 \qquad \gamma(t) = e^{it} \qquad t \in [\pi/2, 3\pi/2]$

$\gamma'(t) = i \cdot e^{it}$



$$\int_{\pi/2}^{3\pi/2} e^{2it} \cdot i e^{it} \, dt = \int_{\pi/2}^{3\pi/2} i e^{3it} \, dt = i\,\frac{e^{3it}}{3i}\Bigg|_{\pi/2}^{3\pi/2} = \frac{1}{3}\left[e^{i\,9\pi/2} - e^{i\,3\pi/2}\right]$$

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